Anatolij Dvurečenskij
Omid Zahiri

Some Results on Quasi MV-Algebras and Perfect Quasi MV-Algebras

Abstract. Quasi MV-algebras are a generalization of MV-algebras and they are motivated by the investigation of the structure of quantum logical gates. In the first part, we present relationships between ideals, weak ideals, congruences, and perfectness within MV-algebras and quasi MV-algebras, respectively. To achieve this goal, we provide a comprehensive characterization of congruence relations of a quasi MV-algebra $\mathcal{A}$ concerning the congruence relations of its MV-algebra of regular elements of $\mathcal{A}$, along with specific equivalence relations concerning the complement of the set of regular elements. In the second part, we concentrate on perfect quasi MV-algebras. We present their representation by symmetric quasi ℓ-groups, a special kind of quasi ℓ-groups. Moreover, we establish a categorical equivalence of the category of perfect quasi MV-algebras, the category of n-perfect quasi MV-algebras, and the category of symmetric quasi ℓ-groups.

Keywords: MV-algebra, Quasi MV-algebra, Perfect quasi MV-algebra, n-perfect quasi MV-algebra, Flat quasi MV-algebra, Weak ideal, Prime congruence, Quasi ℓ-group, Symmetric quasi ℓ-group, Categorical equivalence.

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The paper is dedicated to Prof. Constatine Tsinakis, an outstanding mathematician and friend, on the occasion of his birthday

1. Introduction

Today, the exploration of algebraic structures related to logic is invaluable. These investigations empower us to analyze the associated logic using algebraic tools and facilitate the resolution of intricate problems. Additionally, we can study the logical aspects and also applications of well-known objects

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of these algebraic structures. In this point of view, MV-algebras were introduced by Chang [4] as an algebraic counterpart of infinite-valued Łukasiewicz logic.

Quasi MV-algebras were introduced in [18] as a generalization of MV-algebras. Notably, a quasi MV-algebra, originating from quantum computational logic, serves as an algebraic model characterizing the set of all density operators of the Hilbert space $\mathbb{C}^2$, equipped with an appropriate array of quantum logical gates. In contrast to classical computation, quantum computation [22] permits parallel representation of two atomic information bits. Over the past decade, numerous researchers have devoted their efforts to studying and exploring this algebraic structure.

Ledda et al., [18], scrutinized quasi MV-algebras from a purely abstract perspective within the framework of the universal algebra. Their work established that every quasi MV-algebra can be embedded into the direct product of an MV-algebra and a flat quasi MV-algebra. Additionally, they demonstrated a completeness result in relation to a standard quasi MV-algebra over the complex numbers. Giuntini et al., [14], studied a generalization of quasi MV-algebras through the incorporation of a genuine quantum unary operator denoted by $\sqrt{'}$. Algebraic, categorical, and logical aspects of these new structures have been studied in a series of papers [2, 12, 16, 23]. They established the finite model property and the congruence extension property providing semisimple and free algebras descriptions in both the varieties of quasi MV-algebras and $\sqrt{'}$ quasi MV-algebras. Subsequently, they offered representations for quasi MV-algebras utilizing MV-algebras enriched with additional structure. They also verified the lattices of subvarieties of quasi MV-algebras and demonstrated that quasi MV-algebras, along with Cartesian and flat $\sqrt{'}$ quasi MV-algebras, possess the amalgamation property. Kowalski and Paoli in [16] investigated these varieties' structure theory. They supplied a representation of semisimple $\sqrt{'}$ quasi MV-algebras in terms of function algebras. Freytes and Ledda [12] introduced quasi ℓ-groups and established a categorical duality between quasi MV-algebras and quasi ℓ-groups with strong units.

In [3], logics stemming from quasi Wajsberg algebras were investigated. The study included the proof of completeness results for each of these logics along with the classification of deductive filters and reduced matrices. The results were extended for the logic arising from $\sqrt{'}$ quasi MV-algebras in [24].

Chen and Dudek [5, 6], introduced a non-commutative generalization of quantum computational algebras, termed quasi pseudo MV-algebras. They explored fundamental properties and derived essential algebraic results for this new class of algebras, and investigated ideals and congruences in a quasi pseudo MV-algebra. Chen and Davvaz [7] characterized commutative quasi pseudo MV-algebras and examined the relationship between Archimedean quasi pseudo MV-algebras and commutativity. Additionally, they endeavored to classify local and perfect quasi pseudo MV-algebras.

The paper presents new results extending our knowledge about quasi MV-algebras. The aims of our research are:

- Investigation of weak ideals on quasi MV-algebras.
- Characterization and decomposition of (prime) congruences on quasi MV-algebras.
- Relationship between ideals of a quasi MV-algebra $\mathcal{A}$ and congruences on the MV-algebra $R(\mathcal{A})$ of regular elements of $\mathcal{A}$.
- Study of perfect quasi MV-algebras.
- Representation of perfect quasi MV-algebras.
- Categorical equivalence of perfect quasi MV-algebras.
- Categorical equivalence of n-perfect quasi MV-algebras, $n \geq 1$.

This paper is organized as follows: Sect. 3 focuses on investigating (prime) congruences and weak ideals, departing from the conventional study of (prime) ideals and ideal congruences in the general case. We endeavor to establish relationships between congruences and ideal congruences in quasi MV-algebras. In addition, we show that each congruence relation of a quasi MV-algebra $\mathcal{A}$ decomposes into three relations θ_1, θ_2 and θ_3, where the first one is a congruence relation of the MV-algebra $R(\mathcal{A})$, the second one is an equivalence relation and the third one is a type of connection between θ_1 and θ_2 with some special properties. Every such relation provides a congruence relation on $\mathcal{A}$. Specifically, we demonstrate the existence of a one-to-one correspondence between the set of all ideals of $\mathcal{A}$ and the set of ideals of the MV-algebra $R(\mathcal{A})$. Additionally, we characterize weak ideals of a quasi MV-algebra $\mathcal{A}$ through the lens of ideals in the MV-algebra $R(\mathcal{A})$.

Section 4 extends the investigation of perfect quasi MV-algebras, building upon the groundwork laid in [7]. Our focus centers on exploring the connection between a perfect quasi MV-algebra $\mathcal{A}$ and its corresponding MV-algebra $R(\mathcal{A})$. Our analysis establishes that $\mathcal{A}$ achieves perfection if and only if $R(\mathcal{A})$ is a perfect MV-algebra. This crucial insight guides us in deriving a representation of perfect quasi MV-algebras. The categorical equivalence of the category of perfect quasi MV-algebras with a special category of symmetric quasi ℓ-groups is presented in Sect. 5. In Section 6, we study n-perfect quasi MV-algebras together with the categorical equivalence.

2. Preliminaries

This section encompasses fundamental concepts about quasi MV-algebras, which will be employed in subsequent sections.

MV-algebras have been introduced in [4]. Today, we use the following simple axioms: An algebra $\mathcal{M} = (M; \oplus, ', 0, 1)$ of type $(2, 1, 0, 0)$ is an *MV-algebra* provided that $(M; \oplus, 0)$ is a commutative monoid with the neutral element 0 and for all $x, y \in M$:

(i) $x'' = x$;

(ii) $x \oplus 1 = 1$;

(iii) $x \oplus (x \oplus y')' = y \oplus (y \oplus x')'$.

There is a well-known one-to-one relation between MV-algebras and unital Abelian ℓ-groups (G, u) with a fixed strong unit u of G, [21]: Let $\Gamma(G, u)$ be the interval $[0, u]$ in the ℓ-group G. We endow the interval with operations of the truncated sum $x \oplus y = (x + y) \wedge u$ and the negation $x' = u - x$, $x, y \in [0, u]$. Then $\Gamma(G, u) = \{[0, u]; \oplus, ', 0, u)$ is a prototypical example of MV-algebras, with a famous categorical equivalence of Mundici [8,21], $(G, u) \leftrightarrow \Gamma(G, u)$.

Definition 2.1. [18] A *quasi MV-algebra* is an algebraic structure $\mathcal{A} = (A; \oplus', 0, 1)$ of type $(2, 1, 0, 0)$ which satisfies the following identities:

(Q1) $x \oplus (y \oplus z) = x \oplus (z \oplus y)$;

(Q2) $x'' = x$;

(Q3) $x \oplus 1 = 1$;

(Q4) $x \oplus (x \oplus y')' = y \oplus (y \oplus x')'$;

(Q5) $(x \oplus 0)' = x' \oplus 0$;

(Q6) $(x \oplus y) \oplus 0 = x \oplus y$;

(Q7) $0' = 1$.

On a quasi MV-algebra $\mathcal{A}$, three additional binary operations listed below are considered.

$$x \odot y := (x' \oplus y')', \quad x \vee y := x \oplus (x \oplus y')', \quad x \wedge y := (x' \vee y')'.$$

The relation $\leq$ on a quasi MV-algebra $\mathcal{A}$, defined by $x \leq y$ if and only if $x' \oplus y = 1$, is a *pre-order relation* on $\mathcal{A}$, that is, a relation which is reflexive and transitive but not necessarily anti-symmetric. It is not difficult to prove that $x \leq y$ iff $x \wedge y = x \oplus 0$ iff $x \vee y = y \oplus 0$ iff $x' \oplus y = 1$ (see [5,18]). Due to [18, Lem 6], the operation $\oplus$ is associative, commutative, $x \oplus x' = 1$ and $0 \oplus 0 = 0$. Trivially, a quasi MV-algebra $\mathcal{A} = (A; \oplus, ', 0, 1)$ is an MV-algebra iff it satisfies the equation $x \oplus 0 = x$. We define $A \oplus 0 := \{a \oplus 0 \colon a \in A\}$. Then $A \oplus 0 = 0 \oplus A := \{0 \oplus a \colon a \in A\}$, and algebras $\mathcal{A} \oplus 0 = (A \oplus 0; \oplus, ', 0, 1)$ and $0 \oplus \mathcal{A} = (0 \oplus A; \oplus, ', 0, 1)$ are identical MV-algebras.

For any integer $n \in \mathbb{N}$ and any x of a quasi MV-algebra $\mathcal{A}$, we can define $1.x = x$, $2.x = x \oplus x$ and $n.x = (n-1).x \oplus x$, $2 \leq n$. In a dual way, we define x^n. The order of an element $x \in A$, in symbols $\mathrm{ord}(x)$, is the least integer $n \in \mathbb{N}$ such that $n.x = 1$. If no such n exists, then $\mathrm{ord}(x) = \infty$.

Moreover, according to [18], a quasi MV-algebra $\mathcal{A}$ is called: (1) *linear* if it is linear concerning this relation which means that $x \leq y$ or $y \leq x$, for all $x, y \in \mathcal{A}$; (2) *flat* if $0 = 1$. An element $x \in A$ is said to be *regular* if $x = 0 \oplus x$. The set of all regular elements of $\mathcal{A}$ is denoted by $R(\mathcal{A}) = \{x \in A \colon x = 0 \oplus x\}$. According to [18, Lem 15], $R(\mathcal{A})$ is a subalgebra of $\mathcal{A}$ and also an MV-algebra. It is clear that $A \oplus 0 = 0 \oplus A = R(\mathcal{A})$. In addition, a quasi MV-algebra $\mathcal{A}$ is said to be *proper* if $\mathcal{A}$ is not an MV-algebra, equivalently $\leq$ is not a partial order relation, or $R(\mathcal{A})$ is a proper subset of A.

A non-empty subset I of A is an *ideal* of $\mathcal{A}$ if (1) $0 \in I$, (2) I is closed under $\oplus$, and (3) $I = \downarrow I$, where $\downarrow I = \{x \in A \colon x \leq a, \ \exists a \in I\}$. A *weak* ideal is a non-empty subset I of A that satisfies conditions (1), (2), and (3') $x \odot y \in I$ for all $x \in A$ and $y \in I$. The set of all ideals and weak ideals of $\mathcal{A}$ is denoted by $Id(\mathcal{A})$ and $Id_w(\mathcal{A})$, respectively. Clearly, $Id(\mathcal{A}) \subseteq Id_w(\mathcal{A})$ (see [18]).

According to [7], a proper ideal I of a quasi MV-algebra $\mathcal{A}$ is said to be

(i) *maximal* if for each ideal J of $\mathcal{A}$, $I \subseteq J \subseteq A$ implies that $I = J$ or $J = A$,

(ii) *prime* if $J \cap K = I$ implies that $I = J$ or $I = K$ for all ideals J and K of $\mathcal{A}$,

(iii) *perfect* if (1) for each $x, y \in A$ and $n \in \mathbb{N}$, $(x \odot y)^n \in I$ implies that $x^m \in I$ or $y^m \in I$ for some $m \in \mathbb{N}$; (2) for each $x \in A$ there exists $m \in \mathbb{N}$ such that $x^m \in I \Leftrightarrow (x')^n \notin I$ for some $n \in \mathbb{N}$.

Denote by $Max(\mathcal{A})$ and $Spec(\mathcal{A})$ the set of maximal ideals and all prime ideals of $\mathcal{A}$, respectively. Maximal and prime weak ideals are defined similarly to one's ideals.

$\mathcal{A}$ is said to be *local* if it has a unique maximal ideal. A local quasi MV-algebra $\mathcal{A}$ is called *perfect* if for each $x \in A$, $\mathrm{ord}(x) < \infty$ implies that $\mathrm{ord}(x') = \infty$ (see [7]).

If θ is a congruence relation on a quasi MV-algebra $\mathcal{A}$, then for each $x \in A$, the equivalence class of x is denoted by x/θ, and the set $A/\theta = \{x/\theta \colon x \in A\}$, with the operations inherited from $\mathcal{A}$, forms a quasi MV-algebra denoted by $\mathcal{A}/\theta$.

In addition, if I is an ideal of $\mathcal{A}$, then $\theta_I = \{(x, y) \in A^2 \colon x \odot y', y \odot x' \in I\}$ is a congruence relation on $\mathcal{A}$, and $\mathcal{A}/\theta_I$ is simply presented by $\mathcal{A}/I$. A congruence relation θ is said to be an *ideal congruence* if for each $x, y \in A$, $(0 \oplus x, 0 \oplus y) \in \theta$ implies that $(x, y) \in \theta$ (by [18, Sec 3.3], not every congruence is an ideal congruence).

By [18], on each quasi MV-algebra $\mathcal{A}$, the following relations are congruence relations:

$$(a, b) \in \chi \Leftrightarrow a \leq b,\ b \leq a,\ (a, b) \in \tau \Leftrightarrow a = b \text{ or } a, b \in R(\mathcal{A}). \tag{2.1}$$

Furthermore, $\mathcal{A}/\chi$ is an MV-algebra and $\mathcal{A}/\tau$ is a flat quasi MV-algebra (that is, $0 = 1$).

Definition 2.2. [12] A *quasi ℓ-group* is an algebra $(G; +, \vee, \wedge, -, 0)$ of type $(2, 2, 2, 1, 0)$ satisfying the following conditions, where $G + 0 := \{x + 0 \colon x \in G\}$:

(Ql1) $(G + 0; +, \vee, \wedge, -, 0)$ is an Abelian ℓ-group;
(Ql2) $x + (-x) = 0$;
(Ql3) $-(-x) = x$;
(Ql4) $-(x + 0) = -x + 0$;
(Ql5) $x + y = (x + 0) + (y + 0)$;
(Ql6) $x \vee y = (x + 0) \vee (y + 0)$;
(Ql7) $x + (y \vee z) = (x + y) \vee (x + z)$. On each quasi ℓ-group, the relation $x \leq y$ iff $x \wedge y = 0 + x$ is a pre-order relation. In addition, $x + y = y + x$ for all $x, y \in G$, and if we define $|x| = x \vee -x$, $x \in G$, then $|x| = |x + 0|$, see [12, Lem 3.1].

Lemma 2.3. *An algebra $(G; \vee, \wedge, +, -, 0)$ of type $(2, 2, 2, 1, 0)$ is a quasi ℓ-algebra iff it satisfies (Ql1)–(Ql6). Hence, condition (Ql7) in the definition of quasi ℓ-groups is superfluous.*

Proof. Suppose that (Ql1)–(Ql6) hold. We need to prove (Ql7). Choose $x, y, z \in G$. By (Ql1), $(0 + y) \vee (0 + z) \in 0 + G$, so by (Ql6) $y \vee z \in 0 + G$, consequently,

$$0 + (y \vee z) = y \vee z = (0 + y) \vee (0 + z), \qquad \forall y, z \in G. \tag{2.2}$$

It follows from (Ql5) that

$$\begin{aligned} x + (y \vee z) &= (0 + x) + (0 + (y \vee z)), \quad \text{by (Ql5)} \\ &= (0 + x) + ((0 + y) \vee (0 + z)), \quad \text{by (2.2)} \\ &= ((0 + x) + (0 + y)) \vee ((0 + x) + (0 + z)), \quad \text{by (Ql1)} \\ &= (x + y) \vee (x + z), \quad \text{by (Ql5)}. \end{aligned}$$

The proof of the converse is clear. ■

According to [12], a *quasi unit* on a quasi ℓ-group $(G; +, \vee, \wedge, -, 0)$ is a mapping $u : G \to G$ satisfying the following conditions: (i) $0 \leq u(0)$; (ii) $u(x + 0) = u(0) - x$; (iii) if $0 \leq x \leq u(0)$, then $u(0) - u(x) = 0 + x$; (iv) $u(u(x)) = x$. We define $u_0 := u(0)$. Every quasi ℓ-group G admits a quasi unit, e.g. $u(x) := -x$ if $x \neq 0$ and $u(x) = 0$ for $x \neq 0$. If $a \geq 0$, then $u_a(x) = a - x$ is a quasi unit of G.

A quasi unit u on a quasi ℓ-group G is said to be *strong* iff for each $x \in G$, there is an integer $n \geq 0$ such that $|x| \leq nu_0$. A couple (G, u), where G is a quasi ℓ-group with a fixed strong quasi unit u, is called *bounded* iff for each $x \in G \setminus (G + 0)$, we have $-u(0) \leq x \leq 0$ or $0 \leq x \leq u(0)$.

If u is a quasi unit on a quasi ℓ-group G, then $\Gamma_q(0, u) := ([0, u_0]; \oplus, ', 0, u_0)$, where $x \oplus y = u_0 \wedge (x + y)$, $x' = u(x)$, is an orthodox example of quasi MV-algebra, see [12, Prop 2.13]. In addition, [12, Thm 4.7] says that the category whose objects are couples (G, u), where (G, u) is a bounded quasi ℓ-group with is a fixed strong quasi unit u of G, and whose arrows $f : (G_1, u_1) \to (G_2, u_2)$ are homomorphisms of quasi ℓ-groups satisfying $f(u_1(0)) = u_2(0)$, is categorically equivalent to the category of quasi MV-algebras with the functor $(G, u) \mapsto \Gamma_q(G, u)$.

3. Congruences and (Weak) Ideals of Quasi MV-Algebras

Recall that a congruence θ of a quasi MV-algebra $\mathcal{A}$ is an ideal congruence iff $I := 0/\theta$ is an ideal. In addition, $\theta = \theta_I = \{(x, y) \in A \times A \colon x' \odot y, y' \odot x \in I\}$ (see [18, Sec 3.3]). On the other hand, according to [17] for each ideal I of a quasi MV-algebra $\mathcal{A}$, the quotient structure $\mathcal{A}/I$ is an MV-algebra (see also, [6, Thm 3.12]). Hence, θ is an ideal congruence iff $\mathcal{A}/I$ is an MV-algebra. For example, the congruence relation χ defined in (2.1) is an ideal congruence related to the ideal $I = \{x \in A \colon x \leq 0 \leq x\}$. So, ideal congruences are a very special case of congruences on quasi MV-algebras.

In this section, we examine (prime) congruences and weak ideals instead of (prime) ideals and ideal congruences in the general case. We present relations between congruences and ideal congruences of quasi MV-algebras. We show a one-to-one correspondence between the set of all ideals of $\mathcal{A}$ and the set of all ideals of the MV-algebra $R(\mathcal{A})$. We characterize weak ideals of a quasi MV-algebra $\mathcal{A}$ by ideals of the MV-algebra $R(\mathcal{A})$. The results of this section will be used in the next section.

Definition 3.1. A congruence relation $\theta \neq \nabla := A \times A$ of a quasi MV-algebra $\mathcal{A}$ is said to be *prime* if $\mathcal{A}/\theta$ is linear. The set of all prime congruence relations of $\mathcal{A}$ is denoted by $Con_p(\mathcal{A})$.

Example 3.2. Given a proper quasi MV-algebra $\mathcal{A}$, the congruence relation τ from (2.1) is prime, since $\tau \neq \nabla$ and $\mathcal{A}/\tau$ is a flat quasi MV-algebra which is linear.

Furthermore, if $\mathcal{A}$ is linear, then each congruence relation of $\mathcal{A}$ is prime.

We can easily prove that an ideal congruence θ is prime iff $0/\theta$ is a prime ideal in the sense of [6]. In Proposition 3.5 and Corollary 3.6, we will show how we can create a prime congruence on $\mathcal{A}$ using a prime congruence of $R(\mathcal{A})$.

Proposition 3.3. *Let $\mathcal{A}$ be a quasi MV-algebra. Then $\bigcap Con_p(\mathcal{A}) = \Delta$.*

Proof. Let $\mathcal{A}$ be an arbitrary quasi MV-algebra. According to [8, Thm 1.3.3], the intersection of all prime ideals of $R(\mathcal{A})$ is the zero ideal. For each prime ideal I of $R(\mathcal{A})$, θ_I is a congruence relation of the MV-algebra $R(\mathcal{A})$. Set $\overline{\theta}_I := \theta \cup \Delta$, where $\Delta = \{(x,x) : x \in A\}$. We assert $\overline{\theta}_I$ is a congruence on $\mathcal{A}$: For each $(x,y) \in \overline{\theta}_I$ and $a \in A$ by (Q6), we have $(x \oplus a, y \oplus a) = ((0 \oplus x) \oplus (a \oplus a), (0 \oplus y) \oplus (0 \oplus a)) \in \theta_I \subseteq \overline{\theta}_I$. By definition of $\overline{\theta}_I$, $x, y \in R(\mathcal{A})$ or $x = y$. If $x, y \in R(\mathcal{A})$, then $x', y' \in R(\mathcal{A})$ and $(x', y') \in \theta_I \subseteq \overline{\theta}_I$. If $x = y$, then clearly, $(x', y') = (x', x') \in \overline{\theta}_I$.

Let $x/\overline{\theta}_I, y/\overline{\theta}_I \in \mathcal{A}/\overline{\theta}_I$. Then $(0 \oplus x)/\theta_I \leq (0 \oplus y)/\theta_I$ or $(0 \oplus y)/\theta_I \leq (0 \oplus x)/\theta_I$. Assume that the first one holds. By definition of $\overline{\theta}_I$, $(0 \oplus x)/\overline{\theta}_I \leq (0 \oplus y)/\overline{\theta}_I$, consequently, by [18], $x/\overline{\theta}_I \leq (0 \oplus x)/\overline{\theta}_I \leq (0 \oplus y)/\overline{\theta}_I \leq y/\overline{\theta}_I$. Hence, $\overline{\theta}_I \in Con_p(\mathcal{A})$. Analogously we proceed with the second case.

From [8, Prop 1.2.6] and [8, Thm 1.3.3], it follows that $\bigcap\{\overline{\theta}_I : I$ is a prime ideal of $R(\mathcal{A})\} = \Delta$. Therefore, $\bigcap Con_p(\mathcal{A}) = \Delta$. ■

Remark 3.4. (i) Proposition 3.3 provides another proof for [18, Thm 59], which showed that each quasi MV-algebra is a subdirect product of linear quasi MV-algebras. In addition, it proves that for each pair x, y of distinct elements of a quasi MV-algebra $\mathcal{A}$, there exists a prime congruence relation θ such that $(x, y) \notin \theta$.

(ii) If $\mathcal{A}$ is a proper quasi MV-algebra, then none of the congruence relations $\overline{\theta}_I$, in the proof of Proposition 3.3, is an ideal congruence. So, the intersection of such prime congruence relations $\overline{\theta}_I$ of $\mathcal{A}$ is equal to $\Delta = \{(x,x)\colon x \in A\}$. Consequently, $\mathcal{A}$ is a subdirect product of proper linear quasi MV-algebras.

(iii) $\theta \in Con(\mathcal{A})$ is prime iff $\theta \cap R(\mathcal{A})^2$ is a prime congruence of the MV-algebra $R(\mathcal{A})$. The proof is similar to one of Proposition 3.3.

To make computations easier, for each $a \in A$ and each subset $X \subseteq A$, we define:

$$S_a := \{x \in A\colon x \leq a \leq x\}, \qquad\qquad S_X := \bigcup\{S_x\colon x \in X\}. \tag{3.1}$$

Proposition 3.5. *The following statements hold on each quasi MV-algebra $\mathcal{A}$:*

(i) *For each $I \in Id(\mathcal{A})$, there exists a unique ideal J of $R(\mathcal{A})$ such that $I = \cup_{x\in J} x/\chi = \downarrow J$, where $J = 0 \oplus I$. Moreover, $Id(\mathcal{A}) = \{\downarrow I\colon I$ is an ideal of $R(\mathcal{A})\}$.*

(ii) *For each weak ideal J of $\mathcal{A}$, $0 \oplus J$ is an ideal of $R(\mathcal{A})$. In addition, if I is an ideal of $R(\mathcal{A})$ and $S_I = \bigcup_{a\in 0\oplus J} S_a$, then $I \cup X$ is a weak ideal of $\mathcal{A}$ for each $X \subseteq S_I$.*

(iii) *Let θ be a congruence relation of $R(\mathcal{A})$ and $x \in A \setminus R(\mathcal{A})$. For each $y \in S_{0\oplus x} \setminus R(\mathcal{A})$ the set*

$$\theta_{x,y} := \theta \cup \{(x,y),(y,x),(x',y'),(y',x')\} \cup \Delta, \quad \textit{if } (0\oplus x)' \neq 0 \oplus x,$$
$$\theta_{x,y} := \theta \cup \{(x,y),(y,x),(x',y'),(y',x'),(y,y'),(y',y)\} \cup \Delta, \quad \textit{if } (0\oplus x)' = 0 \oplus x.$$

is a congruence relation on $\mathcal{A}$.

Proof. (i) Let I be an ideal of $\mathcal{A}$. The proof is straightforward by [16, Lem 15]. It suffices to set $J = 0 \oplus I$. In addition, the map $f : Id(R(\mathcal{A})) \to Id(\mathcal{A})$, defined by $f(J) = \downarrow J$, is a bijection preserving the inclusion map.

(ii) The proof follows from [16, Thm 18].

(iii) Similarly to the proof of Proposition 3.3, $\overline{\theta} = \theta \cup \Delta$ is a congruence relation on $\mathcal{A}$. First, assume that $0 \oplus x \neq 0 \oplus x'$ and $y \in S_{0\oplus x}$. By definition, $\theta_{x,y}$ is reflexive and symmetric. In addition, $x' \neq x$ and $y' \neq y$, otherwise, $x = x'$ implies that $(0 \oplus x)' = 0 \oplus x' = 0 \oplus x$ which is absurd. Similarly, $y = y'$ implies that $0 \oplus x = (0 \oplus x)'$, since $y \in S_{0\oplus x}$ and so $0 \oplus x = 0 \oplus y$. It follows that $\theta_{x,y}$ is transitive. Let $(a,b) \in \theta_{x,y}$ and $c \in A$. If $(a,b) \in \overline{\theta}$, then $(a \oplus c, b \oplus c) \in \overline{\theta} \subseteq \theta_{x,y}$. Otherwise, $a,b \in \{x,y\}$ or $a,b \in \{x',y'\}$. In the first case, by (Q6), $(a \oplus c, b \oplus c) = ((0 \oplus x) \oplus (0 \oplus c), (0 \oplus y) \oplus (0 \oplus c)) = ((0 \oplus x) \oplus (0 \oplus c), (0 \oplus x) \oplus (0 \oplus c)) \in \theta \subseteq \theta_{x,y}$. In the second case, by (Q5)

$0 \oplus a = 0 \oplus b = (0 \oplus x)'$ and by (Q6), $(a \oplus c, b \oplus c) = ((0 \oplus a) \oplus (0 \oplus c), (0 \oplus b) \oplus (0 \oplus c)) = ((0 \oplus x)' \oplus (0 \oplus c), (0 \oplus x)' \oplus (0 \oplus c)) \in \theta \subseteq \theta_{x,y}$. Clearly, $\theta_{x,y}$ is compatible with the unary operation $'$. Therefore, θ is a congruence of $\mathcal{A}$. The proof remains valid even when $x' \neq x$ and $y' \neq y$ acknowledge the possibility of $0 \oplus x = (0 \oplus x)'$. In addition, $\theta_{x,y}$ is the least congruence of $\mathcal{A}$ containing θ and (x, y).

Now, let $x' = x$. Then if α is a congruence relation of $\mathcal{A}$ such that $\theta \subseteq \alpha$ and $(x, y) \in \alpha$, then $(y, x') = (y, x) \in \alpha$ and $(x', y') \in \alpha$, which imply $(y, y') \in \alpha$. Similarly to the first part of the current proof, we can show that $\theta_{x,y}$ is a congruence relation of $\mathcal{A}$. ∎

Corollary 3.6. *Consider the notation from Proposition 3.5.*

(i) *For each $x, y \in A$, $\theta_{x,y}$ is an atom in $Con(\mathcal{A})$. In addition, for each $\beta \in Con(\mathcal{A})$, there is a congruence $\theta = \beta \cap R(\mathcal{A})^2$ of $R(\mathcal{A})$ such that β is generated by $\{\theta_{x,y} \colon x, y \in X\}$ for a suitable subset $X \subseteq A \setminus R(\mathcal{A})$.*

(ii) *$\theta_{x,y}$ is a prime congruence of $\mathcal{A}$ if and only if θ is a prime congruence of $R(\mathcal{A})$.*

Proof. (i) It follows from Proposition 3.5.

(ii) It follows from Remark 3.4(iii). ∎

In Proposition 3.5 and Corollary 3.6, we derived atoms of the poset $(Con(\mathcal{A}), \subseteq)$. At the end of this section, we aim to fully characterize congruences of a quasi MV-algebra $\mathcal{A}$ using congruences of the MV-algebra $R(\mathcal{A})$.

Theorem 3.7. *Let $C(\mathcal{A}) = A \setminus R(\mathcal{A})$, θ_1 and θ_2 be relations on A, and a one-to-one map $f : X \to C(\mathcal{A})/\theta_2$, where $X \subseteq R(\mathcal{A})/\theta_1$. We assume the following properties:*

(P1) *$\theta_1 \in Con(R(\mathcal{A}))$.*

(P2) *$\theta_2 \subseteq \bigcup_{a \in R(\mathcal{A})} (S_{a/\theta_1} \times S_{a/\theta_1})$.*

(P3) *θ_2 is an equivalence relation on $C(\mathcal{A})$ such that $\theta_2' = \theta_2$, where $\theta_2' = \{(x', y') \colon (x, y) \in \theta_2\}$.*

(P4) *$X' = X$, $f(x/\theta_1) \in (S_{x/\theta_1})/\theta_2$, f preserves $'$, where we denoted $f(x/\theta_1)$ by y_x/θ_1, and $y_x \in S_{x/\theta_1}$ is a non-regular element of $\mathcal{A}$.*

Then $\theta := \bigcup_{i=1}^3 \theta_i$ is a congruence relation on $\mathcal{A}$, where

$$\theta_3 := \Big(\bigcup_{x/\theta_1 \in X} (x/\theta_1 \times f(x/\theta_1)) \Big) \cup \Big(\bigcup_{x/\theta_1 \in X} (f(x/\theta_1) \times x/\theta_1) \Big)$$

$$= \Big(\bigcup_{x/\theta_1\in X}(x/\theta_1\times y_x/\theta_2)\Big)\cup\Big(\bigcup_{x/\theta_1\in X}(y_x/\theta_2\times x/\theta_1)\Big).$$

Proof. If $x\in R(\mathcal{A})$, then $(x,x)\in\theta_1\subseteq\theta$, and if $x\in C(\mathcal{A})$, then $(x,x)\in\theta_2\subseteq\theta$, implying that θ is reflexive. Also, θ_1, θ_2, and θ_3 are symmetric, and the same holds true for θ.

(i) We note that if $(x,y)\in\theta_3$, there exists $a/\theta_1\in X$ such that $(x,y)\in a/\theta_1\times y_a/\theta_2$ or $(y,x)\in a/\theta_1\times y_a/\theta_2$, where $y_a\in S_{a/\theta_1}\cap C(\mathcal{A})$. In the first case, $x/\theta_1=a/\theta_1$ and $y/\theta_2=y_a/\theta_2$, which means $x/\theta_1\in X$ and $f(x/\theta_1)=y/\theta_2$. Similarly, if $(y,x)\in a/\theta_1\times y_a/\theta_2$, then $y/\theta_1=a/\theta_1\in X$, and $f(y/\theta_1)=y_a/\theta_2=x/\theta_2$. Moreover, by (P2) and (P4), $(y,y_a)\in\theta_2$ implies that $0\oplus y\theta_1 0\oplus y_a\in a/\theta_1$, since $y_a\in S_{a/\theta_1}$, consequently, $y\in S_{a/\theta_1}\setminus R(\mathcal{A})$. Summing up,

$$(x,y)\in\theta_3\Leftrightarrow\Big(x/\theta_1\in X, y\in C(\mathcal{A}), f(x/\theta_1)=y/\theta_2\Big)\ \text{ or }$$
$$\Big(y/\theta_1\in X, x\in C(\mathcal{A}), f(y/\theta_1)=x/\theta_2\Big). \tag{3.2}$$

Choose $(x,y),(y,z)\in\theta$.

(ii) If $(x,y),(y,z)\in\theta_1\cup\theta_2$, then $(x,z)\in\theta_1\cup\theta_2\subseteq\theta$. Note that it is impossible for $(x,y)\in\theta_i$ and $(y,z)\in\theta_j$ to hold simultaneously for distinct elements $i,j\in\{1,2\}$, since $R(\mathcal{A})\cap C(\mathcal{A})=\emptyset$.

(iii) Let $(x,y),(y,z)\in\theta_3$. If $x\in C(\mathcal{A})$, then by (i), $y/\theta_1\in X$, $z\in C(\mathcal{A})$ and $x/\theta_2=f(y/\theta_1)=z/\theta_2$, that is $(x,z)\in\theta_2\subseteq\theta$. Also, $y\in C(\mathcal{A})$ implies that $x/\theta_1, z/\theta_1\in X$ and $f(x/\theta_1)=y/\theta_2=f(z/\theta_1)$, consequently, $x/\theta_1=z/\theta_1$, since f is one-to-one. Hence $(x,z)\in\theta_1\subseteq\theta$.

(iv) Suppose that $(x,y)\in\theta_1$ and $(y,z)\in\theta_3$. Then $y\in R(\mathcal{A})$, so by (i), $z\in C(\mathcal{A})$, $y/\theta_1\in X$, and $z/\theta_2=f(y/\theta_1)=f(x/\theta_1)$. Hence, by (3.2), $(x,z)\in\theta_3\subseteq\theta$. Similarly, if $(y,z)\in\theta_1$ and $(x,y)\in\theta_3$, then $y\in R(\mathcal{A})$, $x\in C(\mathcal{A})$, $f(z/\theta_1)=f(y/\theta_1)=x/\theta_2$, due to (3.2), we get that $(x,z)\in\theta_3\subseteq\theta$.

(v) If $(x,y)\in\theta_2$ and $(y,z)\in\theta_3$, then $y\in C(\mathcal{A})$, so by (i), $z/\theta_1\in X$, and $f(z/\theta_1)=y/\theta_2=x/\theta_2$. Now, (3.2) implies that $(x,z)\in\theta_3\subseteq\theta$. If $(x,y)\in\theta_3$ and $(y,z)\in\theta_2$, then similarly, $y\in C(\mathcal{A})$, $x/\theta_1\in X$, and $f(x/\theta_1)=y/\theta_2=z/\theta_2$. Hence by (3.2), $(x,z)\in\theta_3\subseteq\theta$.

From (ii)–(v), we conclude that θ is transitive.

(vi) If $(x,y)\in\theta_1\cup\theta_2$, then by (P1) and (P3), $(x',y')\in\theta_1\cup\theta_2\subseteq\theta$. Suppose that $(x,y)\in\theta_3$. If $y\in C(\mathcal{A})$, then by (i), $x/\theta_1\in X$ and $f(x/\theta_1)=y/\theta_2$. From (P4) it follows that $f(x'/\theta_1)=f(x/\theta_1)'=y'/\theta_2$. So, by (3.2), $(x',y')\in\theta_3\subseteq\theta$. For the case $x\in C(\mathcal{A})$, in a similar way, we can show that $(x',y')\in\theta_3\subseteq\theta$. Hence θ is compatible with $'$.

(vii) Now, let $(x,y)\in\theta$ and $z\in A$ be given.

(1) If $(x,y)\in\theta_1$, then by (Q6), $(x\oplus z, y\oplus z)=(x\oplus(0\oplus z), y\oplus(0\oplus z))\in\theta_1\subseteq\theta$ (since $0\oplus z\in R(\mathcal{A})$).

(2) If $(x,y)\in\theta_2$, then by (P2), there exists $a\in R(\mathcal{A})$ such that $(x,y)\in S_{a/\theta_1}\times S_{a/\theta_1}$, consequently, $x\in S_{b_1}$ and $y\in S_{b_2}$ for some $b_1,b_2\in a/\theta_1$. Thus, $0\oplus x=(0\oplus b_1)\theta_1(0\oplus a)=a$ and $0\oplus y=(0\oplus b_2)\theta_1(0\oplus a)=a$, consequently, $x\oplus z=0\oplus x\oplus z\theta_1 a\oplus z=0\oplus y\oplus z=y\oplus z$. We have $(x\oplus z, y\oplus x)\subseteq\theta_1\subseteq\theta$. In addition, for each $(x,y)\in\theta_2$, $(0\oplus x, 0\oplus y)\in\theta_1$.

(3) Finally, assume that $(x,y)\in\theta_3$. We use (3.2) again. If $x\in C(\mathcal{A})$, then $y/\theta_1\in X$, and $f(y/\theta_1)=x/\theta_2$. Note that by definition of f, $x\in S_{y/\theta_1}\backslash R(\mathcal{A})$. There is $a\in y/\theta_1$ such that $x\in S_a$, so by (Q6), $x\oplus z=(0\oplus x)\oplus z=a\oplus z\theta_1 y\oplus z$, that means $(x\oplus z, y\oplus z)\in\theta_1\subseteq\theta$. A similar proof works for the case $y\in C(\mathcal{A})$. Therefore, θ is a congruence relation on the quasi MV-algebra $\mathcal{A}$. ■

THEOREM 3.8. *For each congruence relation θ of $\mathcal{A}$, there exist θ_1, θ_2 and a bijection map $f: X\to C(\mathcal{A})/\theta_2$ satisfying the conditions in Theorem 3.7.*

PROOF. Let $\theta\in Con(\mathcal{A})$. Set $\theta_1:=\theta\cap R(\mathcal{A})^2$, $\theta_2:=\theta\cap C(\mathcal{A})^2$, $X=\{x/\theta_1 : x\in R(\mathcal{A}), y\in C(\mathcal{A}), (x,y)\in\theta\}$, and $f(x/\theta_1):=z/\theta_2$, for some $z\in(x/\theta)\setminus R(\mathcal{A})$.

(i) Clearly, $\theta_1\in Con(R(\mathcal{A}))$.

(ii) If $(x,y)\in\theta_2$, then $(0\oplus x, 0\oplus y)\in\theta\cap R(\mathcal{A})^2=\theta_1$, which means $(0\oplus x)/\theta_1=(0\oplus y)/\theta_1$, so $x\in S_{0\oplus x}\subseteq S_{(0\oplus x)/\theta_1}$ and $y\in S_{0\oplus y}\subseteq S_{(0\oplus y)/\theta_1}=S_{(0\oplus x)/\theta_1}$, consequently, $(x,y)\in S_{(0\oplus x)/\theta_1}\times S_{(0\oplus x)/\theta_1}$.

(iii) Clearly, θ_2 is an equivalence relation on $C(\mathcal{A})$. By (Q5), $R(\mathcal{A})'=R(\mathcal{A})$ and $C(\mathcal{A})'=C(\mathcal{A})$, so θ_2 is compatible with $'$.

(iv) Let $x/\theta_1\in X$ for some $x\in R(\mathcal{A})$. Then there exists $y\in C(\mathcal{A})$ such that $(x,y)\in\theta$. From $(x',y')\in\theta$ and $y'\in C(\mathcal{A})$ it follows that $x'/\theta\in X$. We claim that f is well-defined. Clearly, $y\in(x/\theta)\setminus R(\mathcal{A})$. Assume that $z\in(x/\theta)\setminus R(\mathcal{A})$. Then $(x,y),(x,z)\in\theta$ implies that $(z,y)\in\theta\cap R(\mathcal{A})^2=\theta_2$, which means $f(x/\theta_1)=y/\theta_2=z/\theta_2$. Hence, f is well-defined and for each $w\in f(x/\theta_1)$ we have $w/\theta_2=f(x/\theta_1)$ and vice versa. In addition, $f(x'/\theta_1)=y'/\theta_2$, since $(y',x')\in\theta$ and $y'\in C(\mathcal{A})$.

(v) First, note that if $a\in x/\theta_1$ and $b\in f(x/\theta_1)$ for $x/\theta_1\in X$, then $(x,a)\in\theta_1\subseteq\theta$ and $(b,z)\in\theta_2\subseteq\theta$ for some $z\in(x/\theta)\setminus R(\mathcal{A})$. Consequently, from $(b,z),(z,x)\in\theta$, it follows that $(b,x)\in\theta$, $(a,b)\in\theta$ and $x/\theta_1\times f(x/\theta_1)\subseteq\theta$. Similarly, $f(x/\theta_1)\times x\theta_1\subseteq\theta$, that is, $\theta_3\subseteq\theta$.

According to the statement of Theorem 3.7 and (iv), we have

$$\theta_3=\Big(\bigcup_{x/\theta_1\in X} x/\theta_1\times f(x/\theta_1)\Big)\cup\Big(\bigcup_{x/\theta_1\in X} f(x/\theta_1)\times x/\theta_1\Big),$$

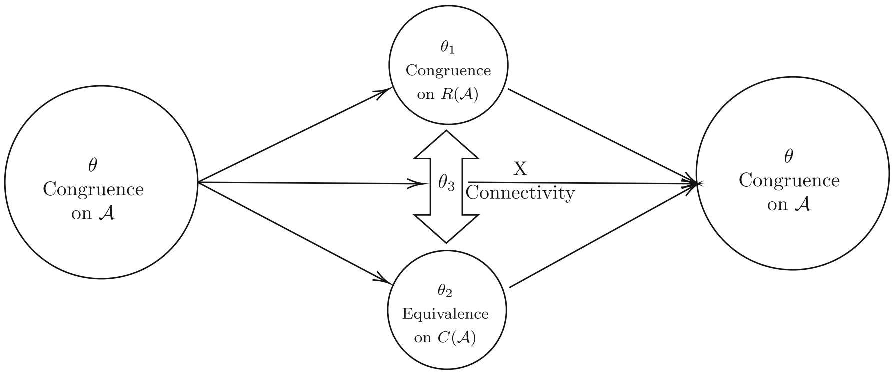

Figure 1. Decomposition of a congruence relation θ on $\mathcal{A}$

$$\begin{aligned} x/\theta_1 \times f(x/\theta_1) &= \{(a,b)\colon a \in x/\theta_1,\ b \in f(x/\theta_1)\ , a \in R(\mathcal{A}), b \in C(\mathcal{A})\} \\ &= \{(a,b)\colon (a,x) \in \theta,\ (x,b) \in \theta\ , a \in R(\mathcal{A}), b \in C(\mathcal{A})\}, \\ f(x/\theta_1) \times x/\theta_1 &= \{(a,b)\colon (b,x) \in \theta,\ (x,a) \in \theta\ , b \in R(\mathcal{A}), a \in C(\mathcal{A})\}. \end{aligned}$$

We have $\theta_3 = \{(a,b) \in \theta\colon (a \in R(\mathcal{A}), b \in C(\mathcal{A}))$ or $(b \in R(\mathcal{A}), a \in C(\mathcal{A}))\} = \theta \setminus (\theta_1 \cup \theta_2)$. Therefore, $\theta = \theta_1 \cup \theta_2 \cup \theta_3$.

(i)–(v) imply that the conditions of Theorem 3.7 hold. ■

The relations θ_1, θ_2, and θ_3 introduced in Theorem 3.7 are denoted as the *regular part*, *flat part*, and *connectivity part* of θ, respectively.

The concepts of maximal and prime ideals in quasi MV-algebras have been explored in previous studies, as detailed in [6, 7]. However, a natural question remains: Can every quasi MV-algebra be associated with maximal (prime) ideals? In the subsequent discussion, we aim to answer this question.

Theorem 3.9. (i) *If $\mathcal{A}$ is a flat quasi MV-algebra, then $\mathcal{A}$ has neither a maximal ideal nor a prime ideal. In addition, $A \setminus \{x\}$ is a maximal (prime) weak ideal for each non-zero element x.*

(ii) *If A is not flat, then $\mathcal{A}$ has a maximal (prime) ideal. In addition, an ideal J of $\mathcal{A}$ is a maximal ideal of $\mathcal{A}$ if and only if $J \cap R(\mathcal{A})$ is a maximal ideal of the MV-algebra $R(\mathcal{A})$. Furthermore, every ideal is included in a maximal ideal.*

(iii) *If I is a maximal weak ideal of $\mathcal{A}$, then $I = \downarrow J$ or $I = A \setminus \{x\}$ for some maximal ideal J of $R(\mathcal{A})$ and some $x \in A \setminus R(\mathcal{A})$.*

Proof. (i) Let I be an ideal of a flat quasi MV-algebra $\mathcal{A}$. Then $1 = 0 \in I$, so $x \leq 1 \in I$ for all $x \in A$ implies that $I = A$. Hence, $Id(\mathcal{A}) = \{A\}$, which means $\mathcal{A}$ does not have any maximal (prime) ideal. Let x be a non-zero element of A, then $0 \in A \setminus \{x\}$, $a \oplus b = 0$, and $a \odot b = 0$ for all $a, b \in A$, so $A \setminus \{x\}$ is a weak ideal of $\mathcal{A}$. It is a maximal weak ideal. Furthermore, if I and J are weak ideals of $\mathcal{A}$ such that $I \cap J \subseteq A \setminus x$, then $x \notin I$ or $x \notin J$, so $I \subseteq A \setminus \{x\}$ or $J \subseteq A \setminus \{x\}$. Therefore, $A \setminus \{x\}$ is a prime weak ideal of $\mathcal{A}$. Now, let J be a maximal ideal of $\mathcal{A}$. By Proposition 3.5(i), $0 \oplus J$ is an ideal of $R(\mathcal{A})$ and $J = \downarrow (0 \oplus J)$. If $H \in Id(R(\mathcal{A}))$ such that $0 \oplus J \subsetneq H \subseteq R(\mathcal{A})$, then there exists $h \in H \setminus 0 \oplus J$ and $J = \downarrow (0 \oplus J) \subseteq \downarrow H \subseteq \mathcal{A}$, in addition, $J = \downarrow (0 \oplus J) \neq \downarrow H$ (otherwise, $h \in 0 \oplus J$). Thus, $\downarrow H = A$, consequently, $R(\mathcal{A}) \subseteq H$. Therefore, $0 \oplus J$ is a maximal ideal of $R(\mathcal{A})$.

(ii) By Proposition 3.5(i), we can show that for each maximal ideal I of $R(\mathcal{A})$, $\downarrow I$ is a maximal ideal of $\mathcal{A}$. Indeed, if $J \in Id(\mathcal{A})$ such that $\downarrow I \subseteq J \subseteq A$, then due to $I = (\downarrow I) \cap R(\mathcal{A}) \subseteq J \cap R(\mathcal{A}) \subseteq R(\mathcal{A})$ and $J \cap R(\mathcal{A}) \in Id(R(\mathcal{A}))$, we get that $J \cap R(\mathcal{A}) = R(\mathcal{A})$ or $J \cap R(\mathcal{A}) = I$. If $J \cap R(\mathcal{A}) = R(\mathcal{A})$, then $J = \downarrow (J \cap R(\mathcal{A})) = \downarrow R(\mathcal{A}) = A$. Similarly, if $J \cap R(\mathcal{A}) = I$, then $J = \downarrow (J \cap R(\mathcal{A})) = \downarrow I$. Therefore, $\downarrow I$ is a maximal ideal of $\mathcal{A}$. Now, [6, Prop 3.5] implies that $\mathcal{A}$ has a prime ideal. Let J be an ideal of $\mathcal{A}$. By [8, Prop 1.2.13], there exists a maximal ideal L of $R(\mathcal{A})$ such that $J \cap R(\mathcal{A}) \subseteq L$. We have $\downarrow L$ is a maximal ideal of $\mathcal{A}$ and $J = \downarrow (J \cap R(\mathcal{A})) \subseteq \downarrow L$.

(iii) Note that if $A \setminus R(\mathcal{A}) = \emptyset$, then $\mathcal{A}$ is an MV-algebra, so by [18], the concept of weak ideals coincides with the concept of ideals, so $\mathcal{A}$ has at least one maximal ideal J, therefore, $J = J \cup (A \setminus R(\mathcal{A}))$ is a maximal weak ideal of $\mathcal{A}$.

Assume that $\mathcal{A}$ is a proper quasi MV-algebra. Take an element $x \in A \setminus R(\mathcal{A})$ and consider the set $I = A \setminus \{x\}$. Then $R(\mathcal{A}) \subseteq I$, and $x \oplus y \in R(\mathcal{A}) \subseteq I$ for all $x, y \in I$. If $x \in A$ and $y \in I$, then $x \odot y \in R(\mathcal{A}) \subseteq I$, so I is a weak ideal of $\mathcal{A}$ which is maximal.

Finally, let K be an arbitrary maximal weak ideal of $\mathcal{A}$. There are two cases:

Case 1. If $R(\mathcal{A}) \subseteq K$, then $\mathcal{A}$ must be proper, so there exists $x \in A \setminus H$ (thus $x \notin R(\mathcal{A})$), consequently, $K \subseteq A \setminus \{x\} \subseteq A$ implies that $K = A \setminus \{x\}$.

Case 2. If $R(\mathcal{A})$ is not a subset of K, then by Proposition 3.5(ii), $0 \oplus K$ is an ideal of $R(\mathcal{A})$. By [8], there is a maximal ideal J of the MV-algebra $R(\mathcal{A})$ such that $0 \oplus K \subseteq J \subsetneq R(\mathcal{A})$. Due to Proposition 3.5(i), $\downarrow (0 \oplus K) \subseteq \downarrow J \subseteq A$. Also, $K \subseteq \downarrow (0 \oplus K)$ (because, by [18, Lem 11], $x \leq 0 \oplus x \in 0 \oplus K$). On the other hand, for each $b \in R(\mathcal{A}) \setminus J$, $b \notin \downarrow J$. So, $K = \downarrow K = \downarrow J$. Note that $\downarrow J$ and $\downarrow (0 \oplus K)$ are ideals (and weak ideals) of $\mathcal{A}$. ■

4. On Perfect Quasi MV-Algebras

In this section, we continue in the study of perfect quasi MV-algebras, building upon the work initiated in [7]. We examine the relationship between a perfect quasi MV-algebra $\mathcal{A}$ and the corresponding MV-algebra $R(\mathcal{A})$. Our demonstration establishes that $\mathcal{A}$ attains perfection if and only if $R(\mathcal{A})$ qualifies as a perfect MV-algebra. This observation guides us to derive a representation of perfect quasi MV-algebras. We recall that the relations τ and χ in the present section were defined by (2.1).

PROPOSITION 4.1. *Let $\mathcal{A}$ be a quasi MV-algebra. Then $\mathcal{A}$ is perfect if and only if $\mathcal{A}/\tau$ is a perfect MV-algebra. In addition, no flat quasi MV-algebra is perfect.*

PROOF. First, note that if $\mathcal{A}$ is a flat quasi MV-algebra, for each $x, y \in A$ we have $1 = (x \oplus y) \oplus 1 = (x \oplus y) \oplus 0 = x \oplus y$. Hence, for each $x \in A$ we have $x \oplus x = 1$ which implies that $\mathrm{ord}(x) < \infty$. Therefore, $\mathcal{A}$ is not perfect.

By [18, Thm 55], the map $f : A \to A/\chi \times A/\tau$, defined by $f(a) = (a/\chi, a/\tau)$, is a subdirect embedding. On the other hand, the mapping $\varphi(x) := x \oplus 0$ is an onto homomorphism from $\mathcal{A}$ to $R(\mathcal{A})$ and $A/\chi = A/\ker(\varphi) \cong R(\mathcal{A})$ (see [18, p. 253]). For each $x \in R(\mathcal{A})$, $\mathrm{ord}(x) < \infty$ implies that $\mathrm{ord}(x') = \infty$, that is $R(\mathcal{A})$ is a perfect MV-algebra. Conversely, assume that $R(\mathcal{A})$ is a perfect MV-algebra. By Theorem 3.9(ii), $\mathcal{A}$ is local. Given $x \in A$ such that $\mathrm{ord}(x) = n$ for some $n \in \mathbb{N}$, we have two cases: If $n = 1$, then $x = 1$ and so $\mathrm{ord}(x') = \mathrm{ord}(0) = \infty$. If $n \geq 2$, by (Q6), $1 = n.x = n.(0 \oplus x)$ which entails $\mathrm{ord}(0 \oplus x)' = \infty$. Thus, $\mathrm{ord}(x') = \infty$, otherwise, $m.x' = 1$ implies that $x' = 1$ (equivalently $x = 0$) or $1 = m.x' = m.(0 \oplus x')$ which is a contradiction in both cases. Therefore, $\mathcal{A}$ is perfect. ■

COROLLARY 4.2. *Let θ be a congruence relation of a quasi MV-algebra $\mathcal{A}$. The following statements hold:*

(i) *If $\mathcal{M} = (M; \oplus, ', 0, 1)$ is a perfect MV-algebra and $\mathcal{A}$ is a flat quasi MV-algebra, the direct product $\mathcal{M} \times \mathcal{A}$ is a perfect quasi MV-algebra.*

(ii) *An ideal I of $\mathcal{A}$ is perfect if and only if $I \cap R(\mathcal{A})$ is a perfect ideal of the MV-algebra $R(\mathcal{A})$.*

(iii) *$\mathcal{A}/\theta$ is perfect if and only if $I = \{x \in R(\mathcal{A}) \colon (0, x) \in \theta\}$ is a perfect ideal of $R(\mathcal{A})$.*

PROOF. (i) Since $\mathcal{M} \cong \mathcal{M} \times \{0\} = R(\mathcal{M} \times \mathcal{A})$, the proof is follows from Proposition 4.1.

(ii) Trivially, $J := I \cap R(\mathcal{A})$ is an ideal of $R(\mathcal{A})$. If I is perfect, due to [7, Prop 5.7], A/I is a perfect quasi MV-algebra. Furthermore, $f : R(\mathcal{A}/I) \to R(\mathcal{A})/J$, defined by $f(a/I) = a/J$, is an isomorphism.

From $a/I \in R(\mathcal{A}/I)$, we get that $0/I \oplus a/I = a/I$, which entails that $a/I = (0 \oplus a)/I$. Thus, each element of $R(\mathcal{A}/I)$ is of the form $(0 \oplus a)/I$ for some $a \in A$. It follows that $a/J \in R(\mathcal{A})/J$. The map f is an onto homomorphism. In addition, $a/I = b/I$ iff $a' \odot b, b' \odot a \in I$ iff $a' \odot b, b' \odot a \in I \cap R(\mathcal{A}) = J$ (by (Q6)) iff $a/J = b/J$ for all $a, b \in R(\mathcal{A})$. Therefore, $R(\mathcal{A}/I) \cong R(\mathcal{A})/J$. Since $\mathcal{A}/I$ is perfect, by Proposition 4.1, $R(\mathcal{A}/I)$ is perfect and so is $R(\mathcal{A})/J$. Hence, by [7, Prop 5.7], J is a perfect ideal of $R(\mathcal{A})$. The proof of the converse follows from $R(\mathcal{A}/I) \cong R(\mathcal{A})/J$ using similar reasonings.

(iii) Let θ be a congruence relation of the quasi MV-algebra $\mathcal{A}$. By Proposition 4.1, $\mathcal{A}/\theta$ is perfect iff $R(\mathcal{A}/\theta)$ is a perfect MV-algebra.

$$R(\mathcal{A}/\theta) = \{0/\theta \oplus a/\theta \colon a \in A\} = \{(0 \oplus a)/\theta \colon a \in A\} = (0 \oplus A)/\theta.$$

The algebra $R(\mathcal{A})/\theta$ is a subalgebra of the quasi MV-algebra $\mathcal{A}/\theta$ which is isomorphic to $R(\mathcal{A})/(\theta \cap R(\mathcal{A})^2) = R(\mathcal{A})/I$. The congruence relation induced by the ideal I in the MV-algebra $R(\mathcal{A})$ is $\theta \cap R(\mathcal{A})^2$. Therefore, $\mathcal{A}/\theta$ is perfect iff I is a perfect ideal of $R(\mathcal{A})$. ■

Since $R(\mathcal{A})$ is isomorphic to $\mathcal{A}/\tau$, from Proposition 4.1, it follows that $\mathcal{A}$ is perfect if and only if $R(\mathcal{A})$ is a perfect MV-algebra.

Proposition 4.3. *Let $\mathcal{A}$ be a quasi MV-algebra. The following statements hold:*

(i) *Let $Rad(\mathcal{A})$ be the intersection of all maximal ideals of $\mathcal{A}$. Then*

$$Rad(\mathcal{A}) = \{x \in A \colon n.x \le x',\ \forall n \in \mathbb{N}\}.$$

(ii) *$B = Rad(\mathcal{A}) \cup Rad(\mathcal{A})'$ is a subalgebra of $\mathcal{A}$ that is a perfect quasi MV-algebra.*

Proof. (i) If $\mathcal{A}$ is flat, then by Theorem 3.9(i), $Max(\mathcal{A}) = \emptyset$, so $Rad(\mathcal{A}) = \bigcap Max(\mathcal{A}) = A$. On the other hand, for each $x \in A$ and each $n \in \mathbb{N}$, $n.x \le 0 \le x'$ (since $\mathcal{A}$ is flat, $x \le y$ for all $x, y \in A$). Hence $Rad(\mathcal{A}) = A = \{x \in A \colon n.x \le x',\ \forall n \in \mathbb{N}\}$.

Now, suppose that $\mathcal{A}$ is not flat. Due to Theorem 3.9(ii), $Max(\mathcal{A}) \neq \emptyset$. Choose $x \in Rad(\mathcal{A})$. Then $x \in J$ for every maximal ideal J of $\mathcal{A}$ and $0 \oplus x \in J \cap R(\mathcal{A})$. From Theorem 3.9(ii), it follows that $0 \oplus x$ belongs to every maximal ideal of the MV-algebra $R(\mathcal{A})$. [8, Prop 3.6.4] implies that $0 \oplus x = 0$ or $n.(0 \oplus x) \le (0 \oplus x)'$ for all $n \in \mathbb{N}$. On the other hand, by (Q6), $n.x = n.(0 \oplus x) \leq x'$ for all integer $n \geq 2$ and by [18, Lem 11(vi)], $x \leq 0 \oplus x \leq x'$. Also, $0 \oplus x = 0$ implies that $x \leq 0 \leq x'$. Hence $x \in \{x \in A\colon n.x \leq x',\ \forall n \in \mathbb{N}\}$. Conversely, if $x \in A$ such that $n.x \leq x'$ for all $n \in \mathbb{N}$. We claim $x \in J$ for every maximal ideal J of $\mathcal{A}$. Choose a maximal ideal J of $\mathcal{A}$. By [18, Lem 11(ii)], $0 \oplus n.x \leq 0 \oplus x' = (0 \oplus x)'$.

Case 1. For each $n \geq 2$ by (Q6), we have $n.(0 \oplus x) = 0 \oplus n.x \leq 0 \oplus x' = (0 \oplus x)'$.

Case 2. For $n = 1$ by [18, Lem 11(vi)], we have $0 \oplus x \leq x \leq 0 \oplus x' = (0 \oplus x)'$.

It follows that $0 \oplus x$ belongs to $Rad(R(\mathcal{A}))$. By [8, Prop 3.6.4] and (i), $0 \oplus x \in J \cap R(\mathcal{A}) \subseteq J$ (note that due to Theorem 3.9, $J \cap R(\mathcal{A})$ is a maximal ideal of $R(\mathcal{A})$). Therefore, $Rad(\mathcal{A}) = \{x \in A\colon n.x \leq x',\ \forall n \in \mathbb{N}\}$.

(ii) First, we show that B is a subalgebra of $\mathcal{A}$. Clearly, $0, 1 \in B$, and B is closed under the unary operation $'$. Choose $x, y \in B$. If $x, y \in Rad(\mathcal{A})$, then $x \oplus y \in Rad(\mathcal{A})$, since it is an ideal of $\mathcal{A}$. If $x, y \in Rad(\mathcal{A})'$, then by [18, Lem 37], $x' \odot y' \in Rad(\mathcal{A})$, so $x \oplus y = (x' \odot y')' \in Rad(\mathcal{A})'$. Now, assume that $x \in Rad(\mathcal{A})$ and $y \in Rad(\mathcal{A})'$. Then $(x \oplus y)' \leq y' \in Rad(\mathcal{A})$, so $(x \oplus y)' \in Rad(\mathcal{A})$, consequently, $x \oplus y \in Rad(\mathcal{A})'$. Hence, B is a subalgebra of $\mathcal{A}$. In addition, $Rad(\mathcal{A})$ is the only maximal ideal of B.

We claim that $(B; \oplus, ', 0, 1)$ is a perfect quasi MV-algebra. Let $x \in B$. If $\mathrm{ord}(x) < \infty$, then $n.x = 1$ for some $n \in \mathbb{N}$ which implies that $x \notin Rad(\mathcal{A})$, by (i). Hence, $x' \in Rad(\mathcal{A})$, which implies that $n.x' \leq x$ for all $n \in \mathbb{N}$. If $x' = 0$, then $\mathrm{ord}(x') = \infty$. Otherwise, $n.x' \leq x < 0' = 1$, so $\mathrm{ord}(x') = \infty$. Therefore, $(B; \oplus, ', 0, 1)$ is a perfect quasi MV-algebra. ■

Let $D(\mathcal{A}) = \{x \in A\colon \mathrm{ord}(x) = \infty\}$, $D^*(\mathcal{A}) = \{x \in A\colon \mathrm{ord}(x) < \infty\}$, and $D_*(\mathcal{A}) = \{x \in A\colon x \geq y',\ \exists y \in D(\mathcal{A})\}$. By [7], $\mathcal{A}$ is perfect iff $D_*(\mathcal{A}) = D^*(\mathcal{A})$. In addition, when $\mathcal{A}$ is perfect, $D(\mathcal{A})$ is the unique maximal ideal of $\mathcal{A}$.

Lemma 4.4. *Let $\mathcal{A}$ be a perfect quasi MV-algebra. Then $D(\mathcal{A}) \odot D(\mathcal{A}) = \{0\}$.*

Proof. Choose $x, y \in D(\mathcal{A})$. Since $\mathcal{A}$ is perfect by Proposition 4.1, $R(\mathcal{A}) \cong \mathcal{A}/\tau$ is perfect. For each $x \in D(\mathcal{A})$ we have $\mathrm{ord}(0 \oplus x) = \infty$, otherwise, $n.(0 \oplus x) = 1$ and (Q6) imply that $n.x = n.(0 \oplus x) = 1$ for $n \geq 2$. Furthermore, if $0 \oplus x = 1$, then $x \oplus x = (0 \oplus x) \oplus x = 1$. Thus, $\mathrm{ord}(0 \oplus x) = \infty$, that is $0 \oplus x \in D(\mathcal{A})$.

Hence $0 \oplus y, 0 \oplus x \in D(R(\mathcal{A}))$. By Proposition 4.1, $R(\mathcal{A})$ is a perfect MV-algebra, and due to [19, Prop 48], we have $(0 \oplus x) \odot (0 \oplus x) = 0$. On the other hand, by [18, Lem 11(vi)] and [5, Lem 3.4(3)], $x \odot y \leq (0 \oplus x) \odot (0 \oplus y) = 0$ and $x \odot y = (0 \oplus x) \odot (0 \oplus y) = 0 \oplus (x \odot y) \in R(\mathcal{A})$, by [5, Lem 3.1(4)]. Consequently, $x \odot y = 0$, since $R(\mathcal{A})$ is an MV-algebra. Therefore, $D(\mathcal{A}) \odot D(\mathcal{A}) = 0$. ■

PROPOSITION 4.5. *The following statements hold on each perfect quasi MV-algebra $\mathcal{A}$:*

(i) $D(\mathcal{A}) \ll D^*(\mathcal{A})$*, that is* $x \leq y$ *for all* $x \in D(\mathcal{A})$ *and* $y \in D^*(\mathcal{A})$.

(ii) *If* $x \oplus a = y \oplus a$ *for* $x, y, a \in D(\mathcal{A})$*, then* $0 \oplus x = 0 \oplus y$.

(iii) $D(\mathcal{A}) = D(R(\mathcal{A})) \cup \{x \in A\colon x \in a/\chi,\ \exists a \in D(R(\mathcal{A}))\}$.

(iv) $D(\mathcal{A})$ *is closed under* $\vee$, $\wedge$, $\oplus$, *and* 0.

PROOF. (i) Choose $x \in D(\mathcal{A})$ and $y \in D^*(\mathcal{A})$. Since $\mathcal{A}$ is perfect and $\mathrm{ord}(y) < \infty$, so $y' \in D(\mathcal{A})$, due to Lemma 4.4, we get $x \odot y' = 0$, consequently, $x \leq y$. Therefore, $D(\mathcal{A}) \ll D^*(\mathcal{A})$.

(ii) (1) Let $x \oplus a = y \oplus a$. Then by (Q6), $(0 \oplus x) \oplus (0 \oplus y) = 0 \oplus (x \oplus a) = 0 \oplus (y \oplus a) = (0 \oplus y) \oplus (0 \oplus a)$.

(2) On the other hand, $0 \oplus a, 0 \oplus x, 0 \oplus y \in D(\mathcal{A})$, since $D(\mathcal{A})$ is an ideal of $\mathcal{A}$. Lemma 4.4 implies that $(0 \oplus x) \odot (0 \oplus a) = 0$ and $(0 \oplus y) \odot (0 \oplus a) = 0$.

We have $0 \oplus x, 0 \oplus y, 0 \oplus a$ belong to the MV-algebra $R(\mathcal{A})$, so (1), (2) and [13, Prop 1.28] imply that $0 \oplus x = 0 \oplus y$.

(iii) The proof is similar to the proof of Proposition 3.5(i).

(iv) Trivially, $0 \in D(\mathcal{A})$. Choose $x, y \in A$. If $x \oplus y \notin D(\mathcal{A})$, then $\mathrm{ord}(x \oplus y) < \infty$, so by [7, Prop 5.1(4)], $\mathrm{ord}(x) < \infty$ or $\mathrm{ord}(y) < \infty$ which are absurd. Hence, $x \oplus y \in D(\mathcal{A})$.

If $\mathrm{ord}(x \vee y) < \infty$, then $\mathrm{ord}(x) = \infty$ and [7, Prop 5.1(4)] implies that $\mathrm{ord}((x \oplus y')') < \infty$, so perfectness of $\mathcal{A}$ implies $\mathrm{ord}(x \oplus y') = \infty$. On the other hand, since $\mathrm{ord}(y) = \infty$, by [7, Cor 5.1], $\mathrm{ord}(y') = n$ for some $n \in \mathbb{N}$, consequently, $n.(x \oplus y') = n.x \oplus n.y' = n.x \oplus 1 = 1$, by (Q3), that is $\mathrm{ord}(x \oplus y') < \infty$. Thus, $x \vee y \in D(\mathcal{A})$.

Similarly, we can show $x \wedge y \in D(\mathcal{A})$. Therefore, $(D(\mathcal{A}); \vee, \wedge, \oplus, 0)$ is a lattice ordered semigroup. ■

EXAMPLE 4.6. Let $(G; \vee, \wedge, +, -, 0)$ be an Abelian ℓ-group. For each $g \in G$ consider a set of symbols X_g satisfying the following two conditions:

(C1) For each $g \in G$, $X_g \cap G = \emptyset$.

(C2) For each $x \in X_g$, there exists a bijection $k_g : X_g \to X_{-g}$ such that $k_g \circ k_{-g}$ and $k_{-g} \circ k_g$ are the identity maps.

Note that X_g can be the empty set; in this case, X_{-g} is the empty set, too. Set $X := \{X_g\colon g \in G\}$ and $G_X := \{\{g, x\}\colon g \in G,\ x \in X_g \cup \{g\}\}$. Define $+ : G_X \times G_X \to G_X$ by $\{g, x\} + \{h, y\} = \{g + h\}$ and $- : G_X \to G_X$ by $-\{g, x\} = \{-g, k_g(x)\}$ for all $g, h \in G$, $x \in X_g$, and $y \in X_h$. Trivially, $+$ and $-$ are well-defined. Given $g, h, k \in G$ and $x \in X_g \cup \{g\}$, $y \in X_h \cup \{h\}$ and $z \in X_k \cup \{k\}$, we have:

(1) $\{0,t\}+\{g,x\}=\{g\}$ for all $t\in X_0\cup\{0\}$.

(2) $\{g,x\}+\{h,y\}=\{g,h\}=\{g\}+\{h\}=\{h,y\}+\{g,x\}$.

(3) $-\{g,x\}+\{g,x\}=\{0\}$ and $-(-\{g,x\})=-\{-g,k_g(x)\}=\{g,k_{-g}\circ k_g(x)\}=\{g,x\}$.

(4) $-(\{g,x\}+\{0\})=-\{g\}=\{-g,-x\}+\{0\}$ and $\{g,x\}+\{h,y\}=\{g+h\}=\{g\}+\{h\}=(\{0\}+\{g,x\})+(\{0\}+\{h,y\})$.

From (1)–(4) it follows that (Ql2)–(Ql5) hold. Now, we define binary operations $\vee$ and $\wedge$ on G_X as follows:

$$\{g,x\}\vee\{h,y\}=\{g\vee h\},\qquad \{g,x\}\wedge\{h,y\}=\{g\wedge h\}. \tag{4.1}$$

We can easily check that $(G_X+\{0\};\vee,\wedge,+,\{0\})$ is an ℓ-group which is isomorphic to G. In addition, (Ql6) and (Ql7) hold. Therefore, $(G_X;\vee,\wedge,+,\{0\})$ is a quasi ℓ-group.

(5) If in the example, $X_g=\emptyset$ for all $g\in G$, then G_X is an ℓ-group and it is isomorphic to G.

Proposition 4.7. *Let G and H be isomorphic ℓ-groups, $f:G\to H$ be an ℓ-group isomorphism, and $X=\{X_g\colon g\in G\}$ satisfy conditions* (C1)–(C2) *of the latter example. For each $h\in H$, set $X'_h=X_g$ if $f(g)=h$, and let $X'=\bigcup\{X'_h\colon h\in H\}$. Then X' satisfies conditions* (C1)–(C2)*, and G_X and $H_{X'}$ constructed in Example 4.6 are isomorphic quasi ℓ-groups.*

Proof. We can assume that $X_g\cap H=\emptyset$ for all $g\in G$ (otherwise, we can use the disjoint union of sets).

For each $h\in H$, we define $X'_h:=X_g$ iff $f(h)=g$. The sets of symbols X'_h $(h\in H)$ satisfy (C1)–(C2). According to Example 4.6, we have

$$\begin{aligned}
G_X&=\Big\{\{g,x\}\colon g\in G,\ x\in X_g\cup\{g\}\Big\},\\
H_{X'}&=\Big\{\{h,x'\}\colon h\in H,\ x'\in X'_{f^{-1}(h)}\cup\{h\}\Big\}\\
&=\Big\{\{f(g),x\}\colon g\in G,\ x\in X_g\cup\{g\}\Big\}.
\end{aligned}$$

Define $f_X:G_X\to H_{X'}$ by $f_X(\{g,x\})=\{f(g),x\}$ and $f_X(\{g\})=\{f(g)\}$, where $g\in G$ and $x\in X_g$. Clearly, $f_X(\{0\})=\{f(0)\}=\{0\}$. Let $\{g,x\}$, $\{h,y\}\in G_X$ for some $g,h\in G$, $x\in X_g\cup\{g\}$ and $y\in X_h\cup\{h\}$. Then $f_X(-\{g,x\})=f_X(\{-g,k_g(x)\})=\{f(-g),k_g(x)\}=\{-f(g),k_g(x)\}=-\{f(g),x\}$ (note that $X_{f(g)}=X_g$ and $X_{-f(g)}=X_{-g}$). Moreover, $f_X(-\{g\})=f_X(\{-g\})=\{f(-g)\}=\{-f(g)\}=-f_X(\{g\})$ and $f_X(\{g,x\}+\{h,y\})=f_X(\{g+h\})=\{f(g+h)\}=f_X(\{g,x\})+f_X(\{h,y\})$. Moreover, according to (4.1), $f_X(\{g,x\}\vee\{h,y\})=f_X(\{g\vee h\})=\{f(g\vee h)\}=\{f(g)\vee f(h)\}=f_X(\{g,x\})\vee f_X(\{h,y\})$. Similarly, f_X preserves $\wedge$ and 1. Therefore, f_X is a homomorphism. The proof of injectivity and surjectivity of f_X is straightforward by definition of f_X. Whence, $G_X \cong H_{X'}$. ■

Due to the latter result, without loss of generality, we can write $H_X := H_{X'}$, so that $G_X \cong H_X$.

PROPOSITION 4.8. *Every quasi ℓ-group is isomorphic to a quasi ℓ-group of the form G_X in Example 4.6.*

PROOF. Let $(G; \vee, \wedge, +, -, 0)$ be a quasi ℓ-group. Consider the ℓ-group $H := G + 0$. For each $h \in H$ set $X_h := \{x \in G\colon 0 + x = 0 + h\} \setminus \{0 + h\}$. If $0+x = 0+h$, then by (Ql4), $0+-x = -(0+x) = -(0+h) = 0+-h$, that is $-x \in X_{-h}$. Thus, for each $x \in X_h$, there exists a unique element $-x \in X_{-h}$ and vice versa. So, a map $k_g : X_g \to X_{-g}$, defined by $k_g(x) = -x$, satisfies (C2). Let $g \in G$. Then $h := 0 + g \in H$ is the only element of H such that $0 + g = 0 + h$. Define $f(g) = \{0 + g, g\}$. Then $f(0) = \{0\}$, $f : G \to H_X$ is well-defined, and onto, evidently. If $g, h \in G$ such that $f(g) = f(h)$, then $\{g, 0 + g\} = \{h, 0 + h\}$.

We have two cases: (1) If $0+g = g$, then $|\{g, 0+g\}| = 1$, so $|\{h, 0+h\}| = 1$ and $h = g$. (2) If $0 + g \neq g$, then $|\{g, 0 + g\}| = 2$, thus $h \neq 0 + h$. We get $0 + g, 0 + h \in H$ and $g, h \notin H$, so $g = h$. That is, f is one-to-one and every element of H_X is uniquely determined by $\{0 + g, g\}$ for some $g \in G$. By (4.1), we have

$$\{0+g, g\} \vee \{0+h, h\} = \{(0+g) \vee (0+h)\} = \{g \vee h\}, \quad \text{by [12, Prop 2.7(10)]} \tag{4.2}$$

$$\{0+g, g\} \wedge \{0+h, h\} = \{(0+g) \wedge (0+h)\} = \{g \wedge h\}, \quad \text{by [12, Prop 2.7(11)].} \tag{4.3}$$

(i) By (Ql6) and (4.2), H_X is a quasi ℓ-group, $f(g) \vee f(h) = \{0 + g, g\} \vee \{0 + h, h\} = \{g \vee h\} = \{g \vee h, 0 + (g \vee h)\} = f(g \vee h)$. Note that $g \vee h \in H$, so $0 + (g \vee h) = g \vee h$. In a similar way, using [12, Prop 2.7] and (4.3), we can show that f preserves $\wedge$.

(ii) From (Ql5), it follows that f preserves $+$.

(iii) $f(-g) = \{(0 + -g), -g\} = \{-(0 + g), -g\} = -\{0 + g, g\} = -f(g)$.

Therefore, f is an isomorphism of quasi ℓ-groups. ■

Assume that $(G; \vee, \wedge, +, -, 0)$ is a quasi ℓ-group and $(H; \vee, \wedge, +, -, 0)$ is a linear quasi ℓ-group. By (Ql1), $0 + G$ is an ℓ-group and $0 + H$ is a linearly ordered group, so the lexicographic product $(0+H) \overrightarrow{\times} (0+G)$ is an ℓ-group.

Consider the following operations on $H \times G$ for all $(h_1, g_1), (h_2, g_2), (h_3, g_3) \in H \times G$:

$$(h_1, g_1) + (h_2, g_2) = (h_1 + h_2, g_1 + g_2), \tag{4.4}$$

$$-(h_1, g_1) = (-h_1, -g_1), \tag{4.5}$$
$$(h_1, g_1) \vee (h_2, g_2) = (0 + h_1, 0 + g_1) \vee (0 + h_2, 0 + g_2), \tag{4.6}$$
$$(h_1, g_1) \wedge (h_2, g_2) = (0 + h_1, 0 + g_1) \wedge (0 + h_2, 0 + g_2). \tag{4.7}$$

Then $(H \overrightarrow{\times} G; \vee, \wedge, +, -, (0,0))$ is a quasi ℓ-group. Indeed:

(i) Clearly, $(0,0) + H \overrightarrow{\times} G$ is the ℓ-group $(0 + H) \overrightarrow{\times} (0 + G)$. So, (Ql1) holds.

(ii) By definition, (Ql2)–(Ql5) hold, trivially.

(iii) According to the definition of $\wedge$ and $\vee$ in (4.4), (Ql6) holds.

Therefore, by Lemma 2.3, $H \overrightarrow{\times} G$ is a quasi ℓ-group.

$$(h_1, g_1) \le (h_2, g_2) \Leftrightarrow (h_1, g_1) \wedge (h_2, g_2) = (0,0) + (h_1, g_1) = (0 + h_1, 0 + g_1) \tag{4.8}$$
$$\Leftrightarrow 0 + h_1 < 0 + h_2 \ \text{ or } \ (0 + h_1 = 0 + h_2 \ \&\ 0 + g_1 \le 0 + g_2). \tag{4.9}$$

Remark 4.9. Assume that $(G; \vee, \wedge, +, -, 0)$ and $(H; \vee, \wedge, +, -, 0)$ are quasi ℓ-groups such that $H \overrightarrow{\times} G$ is a quasi ℓ-group. Then $(0,0) + (H \overrightarrow{\times} G)$ is an ℓ-group, so $(0 + H) \overrightarrow{\times} (0 + G)$ is an ℓ-group, consequently, by [15, Exm 3], $0 + H$ is a linearly ordered group. Given $x, y \in H$, we have $0 + x \le 0 + y$ or $0 + y \le 0 + x$. Suppose that $0 + x \le 0 + y$. Then by [12, Prop 2.9(3)], $x \le 0 + x \le 0 + y \le y$. Therefore, $(H; \vee, \wedge, +, -, 0)$ is linear.

Example 4.10. Let $(G; \vee, \wedge, +, -, 0)$ be a quasi ℓ-group. Consider the quasi ℓ-group $\mathbb{Z} \overrightarrow{\times} G$. Define $u^{lex} := u^{lex}(G) : \mathbb{Z} \overrightarrow{\times} G \to \mathbb{Z} \overrightarrow{\times} G$ by $u^{lex}(x, y) = (1 - x, -y)$. We claim that u^{lex} is a strong quasi unit on $\mathbb{Z} \overrightarrow{\times} G$. Indeed, we have:

(i) $u^{lex}(0,0) = (1,0) \ge (0,0)$ and $u^{lex}(u^{lex}(x,y)) = (x,y)$ for all $x, y \in \mathbb{Z} \overrightarrow{\times} G$.

(ii) $u^{lex}((x,y) + (0,0)) = u^{lex}(x, y + 0) = (1 - x, -(y + 0)) = (1 - x, -y + 0) = (1,0) + (-x, -y) = u(0) - (x,y)$.

(iii) Now, let $(0,0) \le (x,y) \le u^{lex}(0,0) = (1,0)$. Then by (4.8), $(x,y) \in \{0\} \times G^+$ or $(x,y) \in \{1\} \times -G^+$. We have $u^{lex}(0,0) - u^{lex}(x,y) = (1,0) - (1 - x, -y) = (x, 0 + y) = (0,0) + (x,y)$.

(iv) Let $(x,y) \in \mathbb{Z} \overrightarrow{\times} G$. Then $|(x,y)| = |(x,y) + (0,0)| = |(x, y + 0)| \le n(1,0) = nu_0^{lex}$ for some integer $n \ge 0$ (because $u_0^{lex} = (1,0)$ is a strong unit in the ℓ-group $\mathbb{Z} \overrightarrow{\times} (G + 0)$).

(i)–(iv) imply that u^{lex} is a strong quasi unit. Thus, according to [12, Prop 2.13], $\mathcal{A} = (A; \oplus, ', 0, 1) := \Gamma_q(\mathbb{Z} \overrightarrow{\times} G, u^{lex})$ with the interval $[(0,0), (1,0)] = [(0,1), u_0^{lex}]$ is a quasi MV-algebra, where $x \oplus y = (x + y) \wedge u^{lex}$ and $x' =$

$u^{lex}(x)$ for all $x \in \mathbb{Z} \overrightarrow{\times} G$.

$$\begin{aligned}
(0,g) \oplus (0,h) &= (0,g+h) \wedge (1,0) = (0,0+(g+h)), &&\text{if } g,h \in G^+\\
(1,g) \oplus (1,h) &= (2,g+h) \wedge (1,0) = (1,0), &&\text{if } g,h \in -G^+\\
(0,g) \oplus (1,h) &= (1,g+h) \wedge (1,0) = (1,0 \wedge (g+h)), &&\text{if } g,-h \in G^+.
\end{aligned}$$

Given $(x,y) \in A$, $(0,0) \oplus (x,y) = (x, 0+y) \wedge (1,0)$. If $x = 0$, then by (4.4), $(x,0+y) \wedge (1,0) = (x,0+y)$. If $x = 1$, then $y \in -G^+$, again by (4.4), $(x,0+y) \wedge (1,0) = (x,0+y)$. Hence $(0,0) \oplus A = (\{0\} \times (0+G)^+) \cup (\{1\} \times -(0+G)^+) = \Gamma(\mathbb{Z} \overrightarrow{\times} (0+G),(1,0))$ (note that $0+G$ is an ℓ-group). It follows from [1, Thm 9] that $(0,0) \oplus \mathcal{A}$ is a perfect MV-algebra, so by Proposition 4.1, $\mathcal{A}$ is a perfect quasi MV-algebra.

5. Categorical Representation of Perfect Quasi MV-Algebras

In the section, we present a representation of perfect quasi MV-algebras in the form $\Gamma_q(\mathbb{Z} \overrightarrow{\times} G, u^{lex}(G))$, where G is a symmetric quasi ℓ-group. Moreover, we show that the category of perfect quasi MV-algebras is categorically equivalent to the category of symmetric quasi ℓ-groups, which generalizes an analogous result for perfect MV-algebras, see [10]. Introducing the notion of a symmetric quasi ℓ-group is essential for this aim.

We will show that each perfect quasi MV-algebra is isomorphic to $\Gamma_q(\mathbb{Z} \overrightarrow{\times} G, u^{lex})$, where G is a quasi ℓ-group, u^{lex} is the quasi unit defined in Example 4.10 by $u^{lex}(x,y) = (1-x,-y)$, and $u_0^{lex} := u^{leq}(0,0) = (1,0)$. To prove that, we start with the following two lemmas.

Lemma 5.1. *Let $(A;\oplus,',0,1)$ and $(B;\oplus,',0,1)$ be quasi MV-algebras and $f : A \to B$ be a map that preserves $'$, 0, and 1. Then f is a homomorphism of quasi MV-algebras if and only if the restriction of f to $R(\mathcal{A})$ maps $R(\mathcal{A})$ into $R(\mathcal{B})$ and is a homomorphism of MV-algebras such that $f(0 \oplus x) = f(0) \oplus f(x)$ for all $x \in A$.*

Proof. If $f : A \to B$ is a homomorphism of quasi MV-algebras, for each $x \in R(\mathcal{A})$, we have $f(x) = f(0 \oplus x) = f(0) \oplus f(x) \in R(\mathcal{B})$. We can easily verify that $f : R(\mathcal{A}) \to R(\mathcal{B})$ is a homomorphism of MV-algebras.

Conversely, choose $x,y \in A$. By (Q1), (Q6), and the assumption, we get $f(x \oplus y) = f((0 \oplus x) \oplus (0 \oplus y)) = f(0 \oplus x) \oplus f(0 \oplus y) = f(0) \oplus f(x) \oplus f(0) \oplus f(y) = f(x) \oplus f(y)$. Therefore, f is a homomorphism of quasi MV-algebras. ■

Lemma 5.2. *Let $\mathcal{A}$ be a perfect quasi MV-algebra, $\mathcal{B}$ a quasi MV-algebra, and $f : A \to B$ be a homomorphism.*

(i) *We have* $f(D(\mathcal{A})) \subseteq D(\mathcal{B})$, *and* $a \in D(\mathcal{A})$ *if and only if* $f(a) \in D(\mathcal{B})$.

(ii) *If* f *is surjective, then* $f(D(\mathcal{A})) = D(\mathcal{B})$.

Proof. (i) If $x \in D(\mathcal{A})$ is such that $f(x) \notin D(\mathcal{B})$, then $\mathrm{ord}(f(x)) < \infty$, so that $n.f(x) = 1 = f(1)$, consequently, $f((n.x)') = f(0)$ and $(n.x)' \in f^{-1}(\{0\}) \subseteq D(\mathcal{A})$ because $f^{-1}(\{0\})$ is an ideal of $\mathcal{A}$, $D(\mathcal{A})$ is a unique maximal ideal of $\mathcal{A}$ (see the note just before Lemma 4.4) and use Theorem 3.9. It follows that $n.x, (n.x)' \in D(\mathcal{A})$, so $1 \in D(\mathcal{A})$ which is absurd. Hence, $f(D(\mathcal{A})) \subseteq D(\mathcal{B})$. Since $\mathrm{ord}(f(a)) \leq \mathrm{ord}(a)$, we have the end of (i).

(ii) Let f be surjective. Take $y \in D(\mathcal{B})$, so that $y = f(x)$ for some $x \in A$. Then $\mathrm{ord}(f(x)) = \infty$ implies $\mathrm{ord}(x) = \infty$ and $x \in D(\mathcal{A})$, which proves the second part of Lemma. ∎

Theorem 5.3. *Let* $\mathcal{A} = (A; \oplus, ', 0, 1)$ *be a perfect quasi MV-algebra.*

(I) *There exists a quasi* ℓ*-group* G *such that* $\mathcal{A} \cong \Gamma_q(\mathbb{Z} \overrightarrow{\times} G, u^{lex}(G))$.

(II) *The quasi* ℓ*-group* G *can be chosen such that* $G = H_X$ *for some* ℓ*-group* H *with* $\mathcal{A} \oplus 0 \cong \Gamma(\mathbb{Z} \overrightarrow{\times} H, (1, 0))$. *In addition, if* K *is a quasi* ℓ*-group such that* $\mathcal{A} \cong \Gamma_q(\mathbb{Z} \overrightarrow{\times} K, u^{lex}(K))$, *then* G *can be embedded in* K *and* $H \cong K + 0$.

Proof. (I) *Existence of* G. Suppose that $\mathcal{A}$ is perfect. By [7, Cor 5.1], we can easily see that $D(\mathcal{A})' = D^*(\mathcal{A})$. Due to Proposition 4.1, $0 \oplus \mathcal{A}$ is a perfect MV-algebra, so [10] implies that $0 \oplus \mathcal{A} \cong \Gamma(\mathbb{Z} \overrightarrow{\times} H, (1, 0))$ for some ℓ-group H. This ℓ-group H is unique up to isomorphism of ℓ-group. Assume that $\alpha : 0 \oplus A \to \Gamma(\mathbb{Z} \overrightarrow{\times} H, (1, 0))$ is an isomorphism of MV-algebras. Since $0 \oplus A$ is a perfect MV-algebra, we have

$$\begin{aligned} \alpha(0 \oplus D(\mathcal{A})) = \alpha(D(0 \oplus \mathcal{A})) = \{0\} \times H^+, \quad \alpha(0 \oplus D(\mathcal{A})') \\ = \alpha(D(0 \oplus \mathcal{A})') = \{1\} \times -H^+. \end{aligned}$$

Assume that $\pi_1 : \mathbb{Z} \times H \to \mathbb{Z}$ and $\pi_2 : \mathbb{Z} \times H \to H$ are the natural projection maps. For each $h \in H$ consider the following subsets of $C(\mathcal{A}) = A \setminus (0 \oplus A)$:

(i) If $h \in H^+$, then $(0, h) \in \Gamma(\mathbb{Z} \overrightarrow{\times} H, (1, 0))$ and $\alpha^{-1}(0, h) \in 0 \oplus A \subseteq A$. Set $X_h := \{a \in C(\mathcal{A}) \colon 0 \oplus a = 0 \oplus \alpha^{-1}(0, h)\} = \{a \in C(\mathcal{A}) \colon 0 \oplus a = \alpha^{-1}(0, h)\}$, since $\alpha^{-1}(0, h) \in 0 \oplus A$. In addition, if $h \in H^+$, then $a \in X_h$ iff $h = \pi_2(\alpha(0 \oplus a))$ and $\pi_1(\alpha(0 \oplus a)) = 0$.

(ii) If $h \in -H^+$, then $(1, h) \in \Gamma(\mathbb{Z} \overrightarrow{\times} H, (1, 0))$ and $\alpha^{-1}(1, h) \in 0 \oplus A \subseteq A$. Set $X_h := \{a \in C(\mathcal{A}) \colon 0 \oplus a = 0 \oplus \alpha^{-1}(1, h)\} = \{a \in C(\mathcal{A}) \colon 0 \oplus a = \alpha^{-1}(1, h)\}$, since $\alpha^{-1}(1, h) \in 0 \oplus A$. Moreover, if $h \in -H^+$, then $a \in X_h$ iff $h = \pi_2(\alpha(0 \oplus a))$ and $\pi_1(\alpha(0 \oplus a)) = 1$.

(iii) If $h \in H \setminus (H^+ \cup -H^+)$, then set $X_h := \emptyset$.

Given $h \in H^+$ and $a \in X_h$, we have $0 \oplus a = \alpha^{-1}(0,h)$. We claim that $a' \in X_{-h}$. Indeed, $(1,-h) \in \Gamma(\mathbb{Z} \overrightarrow{\times} H, (1,0))$ and $\alpha^{-1}(1,-h) = \alpha^{-1}((0,h)') = (\alpha^{-1}(0,h))' = (0 \oplus a)' = 0 \oplus a'$, by (Q5). In a similar way, if $h \in -H^+$, and $a \in X_h$, then $a' \in H_{-h}$. So, the claim holds. Furthermore, when $X_h \neq \emptyset$, $' : X_h \to X_{-h}$ is a bijection. In addition, for $h \in H \setminus (H^+ \cup -H^+)$, $X_h = \emptyset$, thus, conditions (C1) and (C2) in Example 4.6 are satisfied on the set $X = \{X_h : h \in H\}$.

Due to Example 4.6, $H_X = \{\{h,x\} : h \in H,\ x \in X_h \cup \{h\}\}$ is a quasi ℓ-group with $\{0\}$ as the neutral element, consequently, by Example 4.10, $\Gamma_q(\mathbb{Z} \overrightarrow{\times} H_X, u^{lex})$ is a quasi MV-algebra. Note that the quasi unit $u^{lex} : \mathbb{Z} \overrightarrow{\times} H_X \to \mathbb{Z} \overrightarrow{\times} H_X$ is defined by $u^{lex}(a,y) = (1-a,-y)$ for all $(a,y) \in \mathbb{Z} \overrightarrow{\times} H_X$. We prove that $\mathcal{A} \cong \Gamma_q(\mathbb{Z} \overrightarrow{\times} H_X, u^{lex})$. Define $\beta : A \to \Gamma_q(\mathbb{Z} \overrightarrow{\times} H_X, u^{lex})$ by

$$\beta(a) = \begin{cases} \Big(\pi_1(\alpha(0 \oplus a)), \{\pi_2(\alpha(0 \oplus a)), a\}\Big), & \text{if } a \in C(\mathcal{A}) \\ \Big(\pi_1(\alpha(0 \oplus a)), \{\pi_2(\alpha(0 \oplus a))\}\Big), & \text{if } a \in 0 \oplus \mathcal{A}. \end{cases} \tag{5.1}$$

Due to (i) and (ii), β is well-defined.

(iv) Let $\beta(a) = \beta(b)$. If $a \in C(\mathcal{A})$, then $b \in C(\mathcal{A})$ and $\pi_1(\alpha(0 \oplus a)) = \pi_1(\alpha(0 \oplus b))$ and $\{\pi_2(\alpha(0 \oplus a)), a\} = \{\pi_2(\alpha(0 \oplus b)), b\}$. Evidently, $a = b$. If $a \in 0 \oplus A$, then $b \in 0 \oplus A$, which entails that $\pi_i(\alpha(a)) = \pi_i(\alpha(0 \oplus a)) = \pi_i(\alpha(0 \oplus b)) = \pi_i(\alpha(b))$ for $i = 1,2$, so $a = b$ (since α is one-to-one).

This gives $\beta(0) = (0, \{0\})$ and $\beta(1) = (1, \{0\})$ which is the top element in the quasi MV-algebra $\Gamma_q(\mathbb{Z} \overrightarrow{\times} H_X, u^{lex})$.

(v) Choose an arbitrary element $(x,y) \in \Gamma_q(\mathbb{Z} \overrightarrow{\times} H_X, u^{lex}) = (\{0\} \times H_X^+) \cup (\{1\} \times -H_X^+)$ (see Example 4.10 part (iii)). If $x = 0$ and $|y| = 2$, then $y = \{h,a\}$ for some $h \in H$ and $a \in X_h$. By (i), $\pi_2(\alpha(0 \oplus a)) = h$, $\pi_1(\alpha(0 \oplus a)) = 0$ and $h \in H^+$. We have $\beta(a) = \Big(\pi_1(\alpha(0 \oplus a)), \{\pi_2(\alpha(0 \oplus a)), a\}\Big) = (0, \{h,a\})$. If $|y| = 1$, then $(x,y) = (0, \{h\})$ for $h \in H^+$. By definition, $\beta(\alpha^{-1}(0,h)) = (0,h)$. A similar proof works for $x = 1$. Whence, β is onto.

(vi) If $a \in 0 \oplus A$, then by (Q5), $a' \in 0 \oplus A$ and vice versa, entails that $a \in C(\mathcal{A})$ iff $a' \in C(\mathcal{A})$. For the first case, $\beta(a)' = (\pi_1(\alpha(0 \oplus a)), \{\pi_2(\alpha(0 \oplus a))\}) = (1 - \pi_1(\alpha(0 \oplus a)), \{-\pi_2(\alpha(0 \oplus a))\}) = \beta(a')$. Similarly, if $a \in C(\mathcal{A})$, then $\beta(a)' = (\pi_1(\alpha(0 \oplus a)), \{\pi_2(\alpha(0 \oplus a)), a\}) = (1 - \pi_1(\alpha(0 \oplus a)), \{-\pi_2(\alpha(0 \oplus a)), -a\}) = \beta(a')$.

(vii) By (4.4) and (4.2), for each $a, b \in A$, we have $a \oplus b$ belongs to $0 \oplus A$, see (Q6). It follows that $\alpha(a \oplus b) = \alpha(0 \oplus (a \oplus b)) = \alpha((0 \oplus a) \oplus (0 \oplus b)) =$

$\alpha(0 \oplus a) \oplus \alpha(0 \oplus b)$, and so

$$\begin{aligned}\beta(a \oplus b) &= \Big(\pi_1(\alpha(a \oplus b)), \{\pi_2(\alpha(a \oplus b))\}\Big), \quad \text{since } \ a \oplus b \in 0 \oplus A \\ &= \Big(\pi_1(\alpha((0 \oplus a) \oplus (0 \oplus b))), \{\pi_2(\alpha((0 \oplus a) \oplus (0 \oplus b)))\}\Big).\end{aligned}$$

Since $\alpha : 0 \oplus \mathcal{A} \to \Gamma(\mathbb{Z} \overrightarrow{\times} H, (1, 0))$ is an isomorphism, we have $\Big(\pi_1(\alpha(0 \oplus a)), \pi_2(\alpha(0 \oplus a))\Big) \oplus \Big(\pi_1(\alpha(0 \oplus b)), \pi_2(\alpha(0 \oplus b))\Big) = \alpha(0 \oplus a) \oplus \alpha(0 \oplus b) = \alpha(a \oplus b) = \Big(\pi_1(\alpha(a \oplus b)), \pi_2(\alpha(a \oplus b))\Big)$. Thus, $\beta(0 \oplus a) \oplus \beta(0 \oplus b) = \Big(\pi_1(\alpha(0 \oplus a)), \{\pi_2(\alpha(0 \oplus a))\}\Big) \oplus \Big(\pi_1(\alpha(0 \oplus b)), \{\pi_2(\alpha(0 \oplus b))\}\Big) = \Big(\pi_1(\alpha(a \oplus b)), \{\pi_2(\alpha(a \oplus b))\}\Big) = \beta(a \oplus b)$.

If $a \in 0 \oplus A$, then $\beta(a) = \Big(\pi_1(\alpha(0 \oplus a)), \{\pi_2(\alpha(0 \oplus a))\}\Big) \in R$ $(\Gamma_q(\mathbb{Z} \overrightarrow{\times} H_X, u^{lex})) = \{(0, \{h\}) \colon h \in H^+\} \cup \{(1, \{-h\}) \colon h \in H^+\}$. In addition, $(0, \{0\}) \oplus \beta(a) = \Big(\pi_1(\alpha(0 \oplus a)), \{\pi_2(\alpha(0 \oplus a))\}\Big) = \beta(0 \oplus a)$.

Hence, by Lemma 5.1, β is a homomorphism of quasi MV-algebras.

From (iv)–(vii), we conclude that β is an isomorphism and $\mathcal{A} \cong \Gamma_q (\mathbb{Z} \overrightarrow{\times} H_X, u^{lex})$. Consequently, if we set $G := H_X$, we get the first assertion in question.

(II) We prove the second part of the statement. Assume that $(K; +, \vee, \wedge, -, 0)$ is a quasi ℓ-group and $f : \mathcal{A} \to \Gamma_q(\mathbb{Z} \overrightarrow{\times} K, v^{lex})$, where $v^{lex} := u^{lex}(K)$, is an isomorphism. The restriction $f|_{0 \oplus \mathcal{A}} : 0 \oplus \mathcal{A} \to (0, 0) \oplus \Gamma_q(\mathbb{Z} \overrightarrow{\times} K, v^{lex}) = \Gamma(\mathbb{Z} \overrightarrow{\times} (0 + K), (1, 0))$ is an isomorphism of MV-algebras. On the other hand, $0 \oplus \mathcal{A} \cong \Gamma(\mathbb{Z} \overrightarrow{\times} (\{0\} + H_X), (1, \{0\}))$. It follows that $\{0\} + H_X \cong 0 + K$. Therefore, $H \cong \{0\} + H_X = \{\{h\} \colon h \in H\}$ and $H \cong 0 + K$. Assume that $\mu : \{0\} + H_X \to 0 + K$ is the ℓ-group isomorphism which is induced from the MV-isomorphism

$$f \circ \beta^{-1} : \Gamma\Big(\mathbb{Z} \overrightarrow{\times} (\{0\} + H_X), (1, \{0\})\Big) \to \Gamma\Big(\mathbb{Z} \overrightarrow{\times} (0 + K), (1, 0)\Big).$$

That is, $\mu(x) = \pi_2 \circ f \circ \beta^{-1}(0, x)$ for all $x \in (H_X)^+$. Now, define $\overline{\mu} : H_X \to K$ as follows:

$$\overline{\mu}(x) = \begin{cases} \pi_2(f(\beta^{-1}(0, x))), & \text{if } x \in (H_X)^+ \\ -\pi_2(f(\beta^{-1}(0, -x))), & \text{if } -x \in (H_X)^+ \\ \mu(x), & \text{otherwise.} \end{cases}$$

(viii) Clearly, $\overline{\mu}$ is well-defined and $\overline{\mu}(x) = \mu(x)$ for all $x \in \{0\} + H_X$. That is, $\overline{\mu} : \{0\} + H \to 0 + K$ is an ℓ-group isomorphism.

(ix) If $x \in H_X \setminus ((H_X)^+ \cup -(H_X)^+)$, then $-x \in H_X \setminus ((H_X)^+ \cup -(H_X)^+)$. Thus $\overline{\mu}(-x) = \mu(-x) = -\mu(x) = -\overline{\mu}(x)$. By definition, for $x \in (H_X)^+ \cup -(H_X)^+$, we have $\overline{\mu}(-x) = -\overline{\mu}(x)$, so $\overline{\mu}$ preserves the unary operation $-$.

(x) We claim that $\overline{\mu}(\{0\}) + \overline{\mu}(x) = \overline{\mu}(\{0\} + x)$ for all $x \in H_X$. Clearly, $\overline{\mu}(\{0\}) + \overline{\mu}(x) = 0 + \overline{\mu}(x) = \mu(x) = \mu(\{0\} + x) = \overline{\mu}(\{0\} + x)$ for all $x \in H_X \setminus ((H_X)^+ \cup -(H_X)^+)$. If $x \in (H_X)^+$, there exists $a \in A$ such that $\pi_1(\alpha(0 \oplus a)) = 0$, $\pi_2(\alpha(0 \oplus a)) = h$ and $x = \{h, a\}$. Hence $\overline{\mu}(\{0\} + x) = \overline{\mu}(\{h\})$. On the other hand, $\overline{\mu}(\{0\}) + \overline{\mu}(x) = 0 + \pi_2(f(\beta^{-1}(0, x))) = \pi_2(f(0)) + \pi_2(f(a)) = \pi_2(f(0)) \oplus \pi_2(f(a)) = \pi_2(f(0 \oplus a)) = \pi_2(f(\beta^{-1}(0, \{h\}))) = \overline{\mu}(\{h\})$. So, $\overline{\mu}(0) + \overline{\mu}(x) = \overline{\mu}(0 + x)$ for all $x \in H_X$.

(xi) By (x) and (Ql5), we get that $\overline{\mu}(x + y) = \mu(\{0\} + x + \{0\} + y) = \mu(\{0\} + x) + \mu(\{0\} + y) = \overline{\mu}(\{0\} + x) + \overline{\mu}(\{0\} + y) = \overline{\mu}(0) + \overline{\mu}(x) + \overline{\mu}(\{0\}) + \overline{\mu}(y) = \overline{\mu}(x) + \overline{\mu}(y)$.

In a similar way, by (xi) and (Ql6), we get that $\overline{\mu}(x \vee y) = \overline{\mu}(x) \vee \overline{\mu}(y)$. In addition, $\overline{\mu}(x \wedge y) = \overline{\mu}(x) \wedge \overline{\mu}(y)$, since $x \wedge y = -(-x \vee -y)$ for all $x, y \in H_X$. From (viii)–(xi), it follows that $\overline{\mu}$ is a quasi ℓ-group homomorphism.

(xii) Let $x, y \in H_X$ such that $\overline{\mu}(x) = \overline{\mu}(y)$. If $x \in (H_X)^+$, by definition of $\overline{\mu}$ and Lemma 5.2, $\overline{\mu}(x) \in K^+$. If $x \in -(H_X)^+$, then similarly $\overline{\mu}(x) \in -K^+$. Moreover, if $x \notin (H_X)^+ \cup -(H_X)^+$, then $\overline{\mu}(x) = \mu(x) \notin K^+ \cup -K^+$, since μ is an isomorphism. So, $\overline{\mu}(x) = \overline{\mu}(y)$ implies three cases: (1) $x, y \in (H_X)^+$. (2) $x, y \in -(H_X)^+$. (3) $x, y \notin (H_X)^+ \cup -(H_X)^+$.

(1) In this case, by Lemma 5.2, we have $\beta^{-1}(0, y), \beta^{-1}(0, x) \in D(\mathcal{A})$, so $f(\beta^{-1}(0, y)), f(\beta^{-1}(0, x)) \in D(\Gamma_q(\mathbb{Z} \overrightarrow{\times} K, v^{lex})) = \{0\} \times K^+$, that is $\pi_1(f(\beta^{-1}(0, y))) = \pi_1(f(\beta^{-1}(0, x)))$. On the other hand, by the assumption, $\pi_2(f(\beta^{-1}(0, y))) = \overline{\mu}(y) = \overline{\mu}(x) = \pi_2(f(\beta^{-1}(0, x)))$. Thus,

$$\begin{aligned} f(\beta^{-1}(0, y)) &= (\pi_1(f(\beta^{-1}(0, y))), \pi_2(f(\beta^{-1}(0, y)))) \\ &= (\pi_1(f(\beta^{-1}(0, x))), \pi_2(f(\beta^{-1}(0, x)))) = f(\beta^{-1}(0, x)), \end{aligned}$$

which entails $x = y$, since $f \circ \beta^{-1}$ is one-to-one.

(2) The proof of case (2) is similar to (1).

(3) If $x, y \notin (H_X)^+ \cup -(H_X)^+$, then $0 + x = 0 + y$. If $0 + x = 0 + y$, due to the definition of $\overline{\mu}$, we have $\mu(x) = \overline{\mu}(x) = \overline{\mu}(y) = \mu(y)$, consequently $x = y$, since μ is injective. Hence $\overline{\mu}$ is one-to-one.

Furthermore, for each $k \in K^+$ there exists $x \in (H_X)^+$ such that $\overline{\mu}(x) = k$. In fact, given $k \in K^+$, there exists $a \in A$ such that $f(a) = (0, k)$. Since $D(\Gamma_q(\mathbb{Z} \overrightarrow{\times} K, v^{lex})) = \{0\} \times K^+$, by Lemma 5.2, $a \in D(\mathcal{A})$ and $\beta(a) \in \{0\} \times$

$(H_A)^+$. Trivially, for $x := \pi_2(\beta(a))$, we have $\overline{\mu}(x) = \pi_2(f(\beta^{-1}(\beta(a)))) = \pi_2(f(a)) = k$.

As a conclusion of the proof, it is worth noting that if the quasi ℓ-group K satisfies the condition $0+k = k$ for all $k \in K\setminus(K^+\cup -K^+)$, then trivially the map $\overline{\mu}$ becomes onto and so it is an isomorphism. It means that up to isomorphism, there exists a unique quasi ℓ-group K such that $0+k = k$ for all $k \in K \setminus (K^+ \cup -K^+)$ and $\mathcal{A} \cong \Gamma_q(\mathbb{Z} \overrightarrow{\times} K, u^{lex}(K))$. ∎

The following notion will play a central role in the categorical equivalence of perfect quasi MV-algebras.

Definition 5.4. A quasi ℓ-group $(G; +, \vee, \wedge, +, -, 0)$ is *symmetric* if $g = 0 + g$ for each $g \in G \setminus G^+ \cup -G^+$.

For example, every ℓ-group is symmetric. Moreover, if G is an ℓ-group, then the quasi ℓ-group G_X from Example 4.6 is symmetric whenever $X_g = \emptyset$ for all $g \in G\setminus(G^+\cup -G^+)$. The quasi ℓ-group H_X from the proof of Theorem 5.3 has the just mentioned property, and it is symmetric.

In the next lemma, we show that [9, Prop 5.6] holds for symmetric quasi ℓ-groups instead of ℓ-groups.

Lemma 5.5. *Let $(G_i; +, \vee, \wedge, -, 0)$ be a symmetric quasi ℓ-group for $i = 1, 2$ and $f : G_1^+ \to G_2^+$ be a monoid homomorphism that preserves $+$, $\vee$, $\wedge$, and 0. There exists a unique extension of f to a quasi ℓ-group homomorphism $F : G_1 \to G_2$. In addition, we have $F(x) = f(x)$ and $F(-x) = -f(x)$ for all $x \in G_1^+$ and $F(x) = f(x^+) - f(x^-)$, otherwise.*

Proof. Clearly, $f(0 + G_1^+) \subseteq 0 + G_2^+$ so $f_1 := f|_{0+G_1^+} : 0 + G_1^+ \to 0 + G_2^+$ is a monoid homomorphism preserving $\vee$ and $\wedge$. By [9, Prop 5.6], f_1 can be uniquely extended to an ℓ-group homomorphism $F_1 : 0 + G_1 \to 0 + G_2$ by $F_1(x) = f_1(x^+) - f_1(x^-) = f(x^+) - f(x^-)$ for all $x \in 0 + G_1$. Now, choose $g \in G_1$. Define $F : G_1 \to G_2$ as follows:

$$F(g) = \begin{cases} f(g), & \text{if } g \in G_1^+ \\ -f(-g), & \text{if } g \in -G_1^+ \\ F_1(g), & \text{otherwise.} \end{cases}$$

Since G_1 and G_2 are symmetric, F is well-defined and $F|_{G_1^+} = f$. In addition, F preserves $-$ and

$$\begin{aligned} F(x) + F(0) &= f(x) + f(0) = f(x + 0) = F(x + 0), \quad \forall x \in G_1^+, \\ F(x) + F(0) &= -f(-x) + f(0) = -f(-x + 0) = -F_1(-x + 0) = F_1(x + 0) \\ &= F(x + 0), \quad \forall x \in -G_1^+, \end{aligned}$$

$$F(x) + F(0) = F_1(x) + f(0) = F_1(x) + F_1(0) = F_1(x+0) = F(x+0), \quad \forall x \in G_1 \setminus (G_1^+ \cup -G_1^+).$$

Hence, $F(x) + 0 = F(x) + F(0) = F(x+0)$ for all $x \in X$. Using (QL5) and (QL6) for all $x, y \in G_1$, we get

$$\begin{aligned} F(x+y) &= F(x+0+y+0) = F_1(x+0+y+0) = F_1(x+0) + F_1(y+0) \\ &= F(x+0) + F(y+0) = F(x) + F(0) + F(y) + F(0) = F(x) + F(y). \end{aligned}$$

In a similar way, we can show that F preserves $\vee$ and $\wedge$. Therefore, F is a quasi ℓ-group homomorphism. Now, let $F' : G_1 \to G_2$ be a quasi ℓ-group homomorphism such that $F'(x) = f(x)$ for all $x \in G_1^+$. Then $F'|_{0+G_1} : 0 + G_1 \to 0 + G_2$ is an ℓ-group homomorphism and $F'(x) = f(x)$ for all $x \in (0 + G_1^+)$, so by [9, Prop 5.6], $F'|_{0+G_1} = F|_{0+G_1}$. Now, choose $x \in G_1$. Since G_1 is symmetric, $x \in 0+G_1$, or $x \in G_1^+ \cup -G_1^+$. If $x \in G_1^+$, then by the assumption, $F'(x) = f(x) = F(x)$. If $x \in -G_1^+$, then $F'(x) = -F'(-x) = -f(-x) = -F(-x) = F(x)$. If $x \in 0 + G_1$, then $F'(x) = F(x)$, since $F'|_{0+G_1} = F|_{0+G_1}$. ■

Note that the proof of Lemma 5.5 does not work when G_1 is not symmetric. The following corollary is a straight consequence of Lemma 5.5.

COROLLARY 5.6. *If G_1 and G_2 are symmetric quasi ℓ-groups and $f, h : G_1 \to G_2$ are homomorphisms of quasi ℓ-groups such that $f|_{G_1^+} = h|_{G_1^+}$, then $f = h$.*

Let QMV be the category of quasi MV-algebras, whose objects are quasi MV-algebras and morphisms are homomorphisms of quasi MV-algebras. The category PQMV of perfect quasi MV-algebras has objects perfect quasi MV-algebras and morphisms are homomorphisms of perfect quasi MV-algebras. The category QLG of quasi ℓ-groups has objects quasi ℓ-groups and morphisms are homomorphisms of quasi ℓ-groups. Finally, let SQLG be the category of symmetric quasi ℓ-groups whose objects are symmetric quasi ℓ-groups and morphisms are homomorphisms of quasi ℓ-groups. Define a mapping $\mathcal{P} : \mathsf{SQLG} \to \mathsf{PQMV}$ by

$$\mathcal{P}(G) = \Gamma_q(\mathbb{Z} \overrightarrow{\times} G, u^{lex}(G))$$

if G is an object in SQLG and if $h : G_1 \to G_2$ is a morphism of symmetric quasi ℓ-groups, then we define

$$\mathcal{P}(h)(x) = \begin{cases} (0, h(g)), & \text{if } x = (0, g), \\ (1, -h(g)), & \text{if } x = (1, -g), \end{cases} \qquad g \in G_1^+.$$

Lemma 5.7. *The mapping $\mathcal{P}$ is a full and faithful functor from the category* SQLG *of symmetric quasi ℓ-groups into the category* PQMV *of perfect quasi MV-algebras.*

Proof. If G_1 and G_2 are symmetric quasi ℓ-groups and $h : G_1 \to G_2$ is a morphism of quasi ℓ-groups, then $\mathcal{P}(h) : \Gamma_q(\mathbb{Z} \overrightarrow{\times} G_1, u^{lex}(G_1)) \to \Gamma_q(\mathbb{Z} \overrightarrow{\times} G_2, u^{lex}(G_2))$ is a quasi MV-morphism: Let $x = (x_1, x_2) \in \Gamma_q(\mathbb{Z} \overrightarrow{\times} G_1, u^{lex}(G_1))$. If $x = (0, g_1)$ for some $g_1 \in G_1^+$, then $\mathcal{P}(h)(x') = \mathcal{P}(h)(u^{lex}(G_1)(x)) = \mathcal{P}(h)(1, -g_1) = (1, -h(g_1)) = u^{lex}(G_2)(0, h(g_1)) = (0, h(g_1))' = (\mathcal{P}(h)(x))'$. Similarly, we can show that $\mathcal{P}(h)((1, -g_1)') = (\mathcal{P}(h)(1, -g_1))'$.

Recall that $\mathcal{P}(h|_{0+G_1}) : \Gamma(\mathbb{Z} \overrightarrow{\times} (0 + G_1), (1, 0)) \to \Gamma(\mathbb{Z} \overrightarrow{\times} (0 + G_2), (1, 0))$ is a homomorphism of MV-algebras, by [8, Lem 7.4.2]. We have $\mathcal{P}(h)((0, 0) \oplus x) = \mathcal{P}(h)((0, 0) + x) = \mathcal{P}(h)(x_1, 0 + x_2)$.

If $x_1 = 0$, then by (Ql4), $\mathcal{P}(h)(x_1, 0 + x_2) = (0, h(0 + x_2)) = (0, 0 + h(x_2)) = (0, 0) + (x_1, h(x_2)) = \mathcal{P}(h)(0, 0) + \mathcal{P}(h)(x_1, 0 + x_2) = \mathcal{P}(h)(0, 0) \oplus \mathcal{P}(h)(x_1, 0 + x_2)$. If $x_1 = 1$, then $\mathcal{P}(h)(x_1, 0 + x_2) = \mathcal{P}(h)(x_1, -(0 - x_2)) = (x_1, -h(0 - x_2)) = (0, 0 + h(x_2)) = \mathcal{P}(h)(0, 0) \oplus \mathcal{P}(h)(x_1, x_2)$. Hence

$$\mathcal{P}(h)((0, 0) \oplus x) = \mathcal{P}(h)((0, 0)) \oplus \mathcal{P}(h)(x). \tag{5.2}$$

Since $\mathcal{P}(h|_{0+G_1})$ preserves $\oplus$, using (Ql5) and (5.2), we get $\mathcal{P}(h)$ does preserve $\oplus$. Therefore, it is a quasi MV-homomorphism, and consequently, $\mathcal{P}$ is a functor.

To prove that $\mathcal{P}$ is full, assume that $f : \mathcal{P}(G_1) \to \mathcal{P}(G_2)$ is a morphism of quasi MV-algebras and let $g \in G_1^+$. Then $f(0, g) = (0, g')$ and $f(1, -g)) = (1, -g')$ for a unique $g' \in G_2^+$. Define a mapping $h : G_1^+ \to G_2^+$ by $h(g) = g'$ if and only if $f(0, g) = (0, g')$ and $f(1, -g) = (1, -g')$. Then $h(g_1 + g_2) = h(g_1) + h(g_2)$, $h(g_1 \vee g_2) = h(g_1) \vee h(g_2)$ and $h(g_1 \wedge g_2) = h(g_1) \wedge h(g_2)$ if $g_1, g_2 \in G_1^+$. Due to Lemma 5.5, h can be extended to unique a homomorphism $\hat{h} : G_1 \to G_2$. Moreover, $\mathcal{P}(\hat{h}) = f$.

Now, let h_1 and h_2 be two morphisms from a symmetric quasi ℓ-group G_1 into a symmetric quasi ℓ-group G_2 such that $\mathcal{P}(h_1) = \mathcal{P}(h_2)$. Then $(0, h_1(g)) = (0, h_2(g))$ for any $g \in G_1^+$, consequently $h_1 = h_2$ implying $\mathcal{P}$ is faithful. ■

The following theorem shows that the categories PQMV and SQLG are categorically equivalent. It generalizes the analogous result from [10] on the categorical equivalence of perfect MV-algebras and the category of ℓ-groups.

Theorem 5.8. *The category of symmetric quasi ℓ-groups and the category* PQMV *of perfect MV-algebras and the category* SQLG *of symmetric quasi ℓ-groups are categorically equivalent.*

PROOF. Due to Lemma 5.7, the functor $\mathcal{P} : \mathsf{SQLG} \to \mathsf{PQMV}$ is faithful and full. Let $\mathcal{A}$ be a perfect quasi MV-algebra. If we take the quasi ℓ-group $G = H_X$ from the proof of Theorem 5.3, according to the note just after Definition 5.4, then G is symmetric and $\mathcal{P}(G) = \Gamma_q(\mathbb{Z} \overrightarrow{\times} G, u^{lex}(G)) \cong \mathcal{A}$. Applying [20, Thm IV.1], $\mathcal{P}$ defines a categorical equivalence in question. ■

Due to Theorem 5.3 and its proof, for each perfect pseudo MV-algebra $\mathcal{A}$, there is a symmetric quasi ℓ-group $G_{\mathcal{A}} := H_X$ such that $\mathcal{A} \cong \Gamma(\mathbb{Z} \overrightarrow{\times} G_{\mathcal{A}}, u^{lex}(G_{\mathcal{A}})) = \mathcal{P}(G_{\mathcal{A}})$. The mapping $\beta_{\mathcal{A}} := \beta : \mathcal{A} \to \mathcal{P}(G_{\mathcal{A}})$ defined by (5.1) is an isomorphism of quasi MV-algebras. In what follows, we show that the couple $(G_{\mathcal{A}}, \beta_{\mathcal{A}})$ is a universal arrow from $\mathcal{A}$ to $\mathcal{P}$. Thus, let K be a symmetric quasi ℓ-group and $f : \mathcal{P}(K) \to \mathcal{P}(G_{\mathcal{A}})$ be a homomorphism of quasi MV-algebras. Due to Lemma 5.2, we have $\beta(D(\mathcal{A})) = D(\mathcal{P}(G_{\mathcal{A}})) = \{(0, g) \colon g \in G^+_{\mathcal{A}}\}$ and $f(D(\mathcal{A})) \subseteq D(\mathcal{P}(G)) = \{(0, k) \colon k \in K^+\}$. For each $g \in G^+_{\mathcal{A}}$, there are a unique $a \in D(\mathcal{A})$ and a unique $k \in K^+$ such that $\beta_{\mathcal{A}}(a) = (0, g)$ and $f(a) = (0, k)$. Then the mapping $\phi_0 : G^+_{\mathcal{A}} \to K^+$ given by $\phi_0(g) = k$ is correctly defined, and ϕ_0 preserves $+$, $\vee$, and $\wedge$. Due to Lemma 5.5, ϕ_0 can be uniquely extended to a homomorphism $\phi : G_{\mathcal{A}} \to K$. Define a mapping $h : \mathcal{P}(G) \to \mathcal{P}(K)$ by $h(0, g) = (0, \phi_0(g))$ and $h(1, -g) = (1, -\phi_0(g))$ for each $g \in G^+_{\mathcal{A}}$. Then h is a homomorphism of quasi ℓ-groups such that $h \circ \beta_{\mathcal{A}} = f$. Moreover, if $h' : \mathcal{P}(G_{\mathcal{A}}) \to \mathcal{P}(K)$ be such that $h' \circ \beta_{\mathcal{A}} = f$, then $h' = h$ proving $(G_{\mathcal{A}}, \beta_{\mathcal{A}})$ is a universal arrow from $\mathcal{A}$ to $\mathcal{P}$. Moreover, this arrow generalizes note (II) in Theorem 5.3.

6. n-Perfect Quasi MV-Algebras

In the section, we generalize the notion of perfect quasi MV-algebras to n-perfect quasi MV-algebras, $n \geq 1$, that is to quasi MV-algebras that are roughly speaking isomorphic to quasi MV-algebras of the form $\Gamma_q(\frac{1}{n}\mathbb{Z} \overrightarrow{\times} G, u^{lex}(G))$, where G is a quasi ℓ-group. In addition, we present that the category of n-perfect quasi MV-algebras is also categorically equivalent to the category of symmetric quasi ℓ-groups. To prove it, we use ideas from the previous section.

Let $x, y \in A$. Since $x \oplus y = (x \oplus 0) \oplus (y \oplus 0)$, see (Q6), the regular elements $x \oplus 0$, $y \oplus 0$ belong to the MV-algebra $R(\mathcal{A})$. We can define a partial addition $+$ on $\mathcal{A}$ by $x + y := x \oplus y$ iff $(x \oplus 0) \odot (y \oplus 0) = 0$ or equivalently, $x \oplus 0 \leq y' \oplus 0$. Given an element x of a quasi MV-algebra $\mathcal{A}$, we set $0x := 0$, $1x := x$. If $n \geq 2$, we define $nx = (n-1)x + x$ whenever $(n-1)x \oplus 0 \leq x' \oplus 0$.

Definition 6.1. Let $n \geq 1$ be an integer. We say that a quasi MV-algebra $\mathcal{A} = (A; \oplus,', 0, 1)$ is *n-perfect* if there are non-empty and mutually disjoint subsets $A_0, A_1, \ldots, A_n$ of A such that (i) $A_0 \cup \cdots \cup A_n = A$, (ii) $A_i \leq A_j$ whenever $0 \leq i < j \leq n$, that is $x \leq y$ for all $x \in A_i$ and $y \leq A_j$, (iii) $A_i' := \{x' \colon x \in A_i\} = A_{n-i}$, $i = 0, 1, \ldots, n$, and (iv) if $A_i \oplus A_j := \{x \oplus y \colon x \in A_i, y \in A_j\}$, then $A_i \oplus A_j \subseteq A_{\min\{n, i+j\}}$ for $i, j = 0, 1, \ldots, n$. Clearly, in (ii), we have $x < y$ for $x \in A_i$ and $y \in A_j$ with $i < j$.

We note that if $n = 1$, a 1-perfect quasi MV-algebra is exactly a perfect quasi MV-algebra.

Proposition 6.2. *Let $\mathcal{A}$ be a quasi MV-algebra. We have:*

(1) *If $\mathcal{A}$ is an n-perfect quasi MV-algebra, then $R(\mathcal{A})$ is an n-perfect MV-algebra.*

(2) *If $A_0, \ldots, A_n$ are subsets of A satisfying Definition 6.1, then A_0 is a maximal ideal of $\mathcal{A}$.*

(3) *If $A_0, \ldots, A_n$ and $B_0, \ldots, B_n$ are subsets of A satisfying Definition 6.1, then $A_i = B_i$ for each $i = 0, 1, \ldots, n$.*

Proof. (1)–(2) The sets $A_i \cap \mathcal{R}(\mathcal{A})$, $i = 0, 1, \ldots, n$, form a decomposition of the MV-algebra $R(\mathcal{A})$ in the sense of Definition 6.1, so then $R(\mathcal{A})$ is an n-perfect MV-algebra in the sense of [11]. Whence, $A_0 \cap R(\mathcal{A})$ is a maximal ideal of $R(\mathcal{A})$, see [11, Thm 4.1(v)], and by Theorem 3.9(iii), A_0 is a maximal ideal of $\mathcal{A}$.

(3) If subsets $B_0, \ldots, B_n$ of A satisfy the conditions of the definition, then $A_i \cap R(\mathcal{A}) = B_i \cap R(\mathcal{A})$ $(i = 0, 1, \ldots, n)$ due to the uniqueness in n-perfect MV-algebras. In addition, we have $A_i = B_i$ for each $i = 0, 1, \ldots, n$: Choose an arbitrary $x \in A_i$. There exists B_j such that $x \in B_j$. Since $x \oplus 0 \in A_i \oplus A_0 = A_i$, $x \oplus 0 \in B_j$, and $x \oplus 0$ is regular, we have $x \oplus 0 \in A_i \cap R(\mathcal{A}) = B_i \cap R(\mathcal{A})$. On the other hand, $x \in B_j \cap R(\mathcal{A})$, which yields $i = j$ and $A_i \subseteq B_i$. In an analogous way, we can show $B_i \subseteq A_i$. Or we can use that $B_0, \ldots, B_n$ form a disjoint decomposition of A, so in both ways, $A_i = B_i$. ∎

Therefore, by (3) of the latter proposition, we can write unambiguously $\mathcal{A} = (A_0, \ldots, A_n)$ for an n-perfect quasi MV-algebra $\mathcal{A}$.

There is a narrow connection between n-perfect quasi MV-algebra $\mathcal{A}$ and the n-perfect MV-algebra $R(\mathcal{A})$:

Proposition 6.3. *A quasi MV-algebra $\mathcal{A}$ is n-perfect if and only if $R(\mathcal{A})$ is an n-perfect MV-algebra.*

Proof. One direction was proved in Proposition 6.2(1).

Now, assume that $R(\mathcal{A})$ is an n-perfect MV-algebra. Consider subsets $A_0, A_1, \dots, A_n$ of $R(\mathcal{A})$ such that $R(\mathcal{A}) = (A_0, \dots, A_n)$. For each $i \in \{0, 1, \dots, n\}$, we set $B_i := S_{A_i}$ by (3.1).

(i) For each $x \in A$ there exists a unique $i \in \{0, 1, \dots, n\}$ such that $0 \oplus x \in A_i$. It entails that $x \in B_i \subseteq \bigcup_{j=0}^{n} B_j$. In addition, $y \in B_i \cap B_j$ implies that $0 \oplus y \in A_i \cap A_j$, which implies $i = j$. Hence $B_0, B_1, \dots, B_n$ are mutually disjoint.

(ii) Given $0 \leq i < j \leq n$, $x \in B_i$, and $y \in B_j$, we have $x \leq 0 \oplus x \leq 0 \oplus y \leq y$ (use [18, Lem 11(vi)]). Therefore, $B_i \leq B_j$.

(iii) Choose $y \in B_i'$. Then $y' \in B_i$ and by (Q5), we have

$$y' \in B_i \Leftrightarrow 0 \oplus y' \in A_i \Leftrightarrow (0 \oplus y)' \in A_i \Leftrightarrow 0 \oplus y \in A_{n-i} \Leftrightarrow y \in B_{n-i}.$$

(iv) Applying (Q6) for each $x \in B_i$ and $y \in B_j$, we get

$$x \oplus y = (0 \oplus x) \oplus (0 \oplus y) \in A_i \oplus A_j = A_{\min\{n, i+j\}} \subseteq B_{\min\{n, i+j\}},$$

yielding $B_i \oplus B_j \subseteq B_{\min\{n, i+j\}}$. Therefore, $\mathcal{A}$ is an n-perfect quasi MV-algebra. ■

Theorem 6.4. *An n-perfect quasi MV-algebra $\mathcal{A} = (A_0, \dots, A_n)$ is isomorphic to an n-perfect quasi MV-algebra of the form $\Gamma_q(\frac{1}{n}\mathbb{Z} \overrightarrow{\times} G, u^{lex}(G))$ for some symmetric quasi ℓ-group G if and only if, there is an element $a \in A_1$ such that na exists in A and $na = 1$. In such a case, G can be symmetric.*

Proof. For one direction, it is enough to take a quasi MV-algebra Γ_q $(\frac{1}{n}\mathbb{Z} \overrightarrow{\times} G, u^{lex}(G))$, where G is a symmetric quasi ℓ-group. If we set $A_0 = \{(0, g) \colon g \in G^+\}$, $A_n = \{(1, g) \colon g \in -G^+\}$, and for $0 < i < n$ we define $A_i = \{(i/n, g) \colon g \in G\}$, then $\Gamma_q(\frac{1}{n}\mathbb{Z} \overrightarrow{\times} G, u^{lex}(G))$ is n-perfect with a fixed element $a = (1/n, 0) \in A_1$ satisfying $na = (1, 0)$.

Conversely, let $\mathcal{A} = (A_0, \dots, A_n)$ be an n-perfect quasi MV-algebra with a fixed element $a \in A_1$ such that na exists in $\mathcal{A}$ and $na = 1$. Without loss of generality, we can assume that the fixed element a is regular. Due to Proposition 6.3, $R(\mathcal{A})$ is an n-perfect MV-algebra, and the element $a \oplus 0$ satisfies $na = n(a \oplus 0) = 1$. By [11], there is an Abelian ℓ-group H such that $R(\mathcal{A}) \cong \Gamma(\frac{1}{n}\mathbb{Z} \overrightarrow{\times} H, (1, 0))$; let $\alpha : R(\mathcal{A}) \to \Gamma(\frac{1}{n}\mathbb{Z} \overrightarrow{\times} H, (1, 0))$ be the isomorphism. As in the proof of Theorem 5.3, construct the symmetric quasi ℓ-group H_X. The mapping $\beta : \mathcal{A} \to \Gamma_q(\frac{1}{n}\mathbb{Z} \overrightarrow{\times} H_X, u^{lex}(H_X))$, defined by (5.1), is an isomorphism also in this case (we use the same ideas as in Theorem 5.3). If we put $G = H_X$, we get the assertion in question. ■

Given an integer $n \geq 1$, the category PQMV_n of n-perfect quasi MV-algebras has objects couples $(\mathcal{A}, a)$, where $\mathcal{A}$ is an n-perfect MV-algebra with a fixed element $a \in \mathcal{A}$ such that na exists in A and $na = 1$. Morphisms are homomorphisms of perfect quasi MV-algebras preserving the fixed elements a's. Moreover, PQMV_n is a category. If $n = 1$ and $(\mathcal{A}, a) \in \mathsf{PQMV}_1$ is an object, for the fixed element $a \in A_1$, we have $a = 1$. Therefore, PQMV_1 is practically PQMV.

Theorem 6.5. *The category* PQMV_n *is categorically equivalent to the category* SQLG.

Proof. Define a mapping $\mathcal{P}_n$ from the category SQLG into PQMV_n as follows: Let G be a symmetric quasi ℓ-group, then

$$\mathcal{P}_n(G) = \Big(\Gamma_q(\frac{1}{n}\mathbb{Z} \overrightarrow{\times} G, u^{lex}(G)), (1/n, 0)\Big),$$

and if $h : G_1 \to G_2$ is a morphism, then

$$\mathcal{P}_n(h)(x) = \begin{cases} (0, h(g)), & \text{if } x = (0, g), g \in G_1^+ \\ (i/n, h(g^+) - h(g^-)), & \text{if } x = (i/n, g), g \in G_1,\ i = 1, \ldots, n-1, \\ (1, -h(g)), & \text{if } x = (1, -g), g \in G_1^+. \end{cases}$$

Then $\mathcal{P}_n$ is a functor. As in Lemma 5.7, $\mathcal{P}_n$ is full and faithful, and by Theorem 6.4, given an object $(\mathcal{A}, a)$, where $\mathcal{A}$ is an n-perfect quasi MV-algebra and $na = 1$, there is a symmetric quasi ℓ-group G such that $(\mathcal{A}, a) \cong \mathcal{P}_n((\Gamma_q(\frac{1}{n}\mathbb{Z} \overrightarrow{\times} G, u^{lex}(G)), (1/n, 0)))$. Due to [20, Thm IV.1], $\mathcal{P}_n$ defines a categorical equivalence in question. ∎

We have a straightforward consequence of the latter theorem and Theorem 5.8:

Corollary 6.6. *The categories* PQMV_n $(n \geq 1)$, PQMV, *and* SQLG *are categorically equivalent.*

We note that it is possible to show that if $\beta_{\mathcal{A}} := \beta$ and $G_{\mathcal{A}} := H_X$, then $(G_{\mathcal{A}}, \beta_{\mathcal{A}})$ is a universal arrow from $(\mathcal{A}, a)$ to $\mathcal{P}_n$.

In the rest of the paper, we show that the category of perfect (n-perfect) quasi MV-algebras is categorically equivalent to the category of Abelian ℓ-groups or to the category of perfect MV-algebras, compare [10].

Let LG be the category of ℓ-groups whose objects are Abelian ℓ-groups and morphisms are homomorphisms of ℓ-groups. Define a mapping $\mathcal{R}$: $\mathsf{SQLG} \to \mathsf{LG}$ by

$$\mathcal{R}(G) := G + 0, \quad G \in \mathsf{SQLG},$$

and if $h : G_1 \to G_2$ is a morphism of symmetric quasi ℓ-groups G_1, G_2, then $\mathcal{R}(h)$ is the restriction of h onto $G_1 + 0$. Clearly, $\mathcal{R}$ is a functor. Analogously we define the category PMV of perfect MV-algebras and the category of n-perfect MV-algebras PMV_n.

Theorem 6.7. *The category of symmetric quasi ℓ-groups* SQLG *is categorically equivalent to the category of* LG *of Abelian ℓ-groups.*

Proof. *Fullness.* Assume that $h : \mathcal{R}(G_1) = (G_1 + 0) \to \mathcal{R}(G_2) = G_2 + 0$, is a morphism of ℓ-groups. For each $i = 1, 2$, construct the symmetric quasi ℓ-group H^i_X corresponding to G_i by Proposition 4.8, and let $f_i : G_i \to H^i_X$ be the isomorphism defined by $f_i(g_i) = \{0 + g_i, g_i\}$, $g_i \in G_i$, see the proof of Proposition 4.8. If we define a mapping $\hat{h} : H^1_X \to H^2_X$ by $\hat{f}(\{0 + g_1, g_1\}) = \{0+h(g_1), h(g_1)\}$ $(g_1 \in G_1)$, it is possible to show that $\hat{h}$ is a homomorphism. Put $\tilde{h} = f_2^{-1} \circ \hat{h} \circ \tilde{f}_1$, then $\tilde{h}$ is a homomorphism from G_1 to G_2 such that $\mathcal{R}(\tilde{h}) = h$.

Faithfulness. Let $h_1, h_2 : G_1 \to G_2$ be two homomorphisms of symmetric quasi ℓ-groups such that $\mathcal{R}(h_1) = \mathcal{R}(h_2)$. That is, $h_1(g_1 + 0) = h_2(g_1 + 0)$, $g_1 \in G_1$. As in the former paragraph, we can show that $\{0+h_1(g_1), h_1(g_1)\} = \{0 + h_2(g_1), h_2(g_1)\}$, $g_1 \in G_1$, giving $h_1 = h_2$. It means, $\mathcal{R}$ is faithful.

Let G be an Abelian ℓ-group. Since G is trivially a symmetric quasi ℓ-group, then $\mathcal{R}(G) = G + 0 = G$, $\mathcal{R}(G)$ is isomorphic to G. Due to [20, Thm IV.1], $\mathcal{R}$ defines a categorical equivalence. ■

Finally, we have the following corollary

Corollary 6.8. *The categories* PQMV, PQMV_n, SQLG, PMV, PMV_n, *and* LG *are mutually categorically equivalent.*

Proof. It follows from Theorem 5.8, Theorem 6.5, Corollary 6.6, and [10]. ■

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References

[1] Belluce, L.P., A. Di Nola, and B. Gerla, Abelian ℓ-groups with strong unit and perfect MV-algebras, *Order* 25: 387–401, 2008. https://doi.org/10.1007/s11083-008-9103-7

[2] Bou, F., F. Paoli, A. Ledda, and H. Freytes, On some properties of $\sqrt{\prime}$ quasi-MV algebras and quasi-MV algebras. Part II, *Soft Computing* 12(4): 341–352, 2008. https://doi.org/10.1007/s00500-007-0185-8

[3] Bou, F., F. Paoli, A. Ledda, M. Spinks, and R. Giuntini, The logic of quasi-MV algebras, *The Journal of Logic and Computation* 20(2): 619–643, 2010. https://doi.org/10.1093/logcom/exp080

[4] Chang, C.C., Algebraic analysis of many-valued logics, *Transactions of the American Mathematical Society* 88: 467–490, 1958. https://doi.org/10.2307/1993227

[5] Chen, W., and W.A. Dudek, Quantum computational algebra with a non-commutative generalization, *Mathematica Slovaca* 66(1): 19–34, 2016. https://doi.org/10.1515/ms-2015-0112

[6] Chen, W., and W.A. Dudek, *Ideals and congruences in quasi-pseudo-MV algebras*, *Soft Computing* 22: 3879–3889, 2018. https://doi.org/10.1007/s00500-017-2854-6

[7] Chen, W., and B. Davvaz, Some classes of quasi-pseudo-MV algebras, *Logic Journal of the IGPL* 24(5): 655–672, 2016. https://doi.org/10.1093/jigpal/jzw034

[8] Cignoli, R., I. M. L. D'Ottaviano and D. Mundici, *Algebraic Foundations of Many-Valued Reasoning*, Springer Science and Business Media, Dordrecht, 2000. https://doi.org/10.1007/978-94-015-9480-6

[9] Darnel, M. *Theory of Lattice-Ordered Groups*, CRC Press, USA, 1994.

[10] Di Nola, A., and A. Lettieri, Perfect MV-algebras are categorically equivalent to Abelian ℓ-groups, *Studia Logica* 53: 417–432, 1994. https://doi.org/10.1007/BF01057937

[11] Dvurečenskij, A. On n-perfect GMV-algebras, *Journal of Algebra* 319(12): 4921–4946, 2008. https://doi.org/10.1016/j.jalgebra.2008.03.027

[12] Freytes, H., and A. Ledda, Categories of semigroups in quantum computational structures, *Mathematica Slovaca* 59(4): 413–432, 2009. https://doi.org/10.2478/s12175-009-0137-0

[13] Georgescu, G., andA. Iorgulescu, Pseudo MV-algebras, *Multiple-Valued Logics* 6: 193–215, 2001.

[14] Giuntini, R., A. Ledda, and F. Paoli, Expanding quasi-MV algebras by a quantum operator, *Studia Logica* 87: 99–128, 2007. https://doi.org/10.1007/s11225-007-9079-0

[15] Glass, A., and Ch.W. Holland, *Lattice-Ordered Groups: Advances and Techniques*, vol. 48 of *Mathematics and Its Applications*, Springer Dordrecht, 1989.

[16] Kowalski, T., and F. Paoli, On some properties of quasi MV-algebras and $\sqrt{\prime}$ quasi-MV algebras. Part III, *Reports on Mathematical Logic* 45: 161–199, 2010.

[17] Kowalski, T., F. Paoli, and M. Spinks, Quasi-subtractive varieties, *Journal of Symbolic Logic* 76(4): 1261–1286, 2011.

[18] Ledda, A., M. Konig, F. Paoli, and R. Giuntini, MV-algebras and quantum computation, *Studia Logica* 82(2): 245–270, 2006. https://doi.org/10.1007/s11225-006-7202-2

[19] Leuştean, I., Local pseudo MV-algebras, *Soft Computing* 5: 386–395, 2001. https://doi.org/10.1007/s005000100141

[20] Mac Lane, S., *Categories for the Working Mathematician*, Springer-Verlag, New York, Heidelberg, Berlin, 1971.

[21] Mundici, D., Interpretation of AF C*-algebras in Łukasiewicz sentential calculus, *ournal of Functional Analysis* 65: 15–63, 1986. https://doi.org/10.1016/0022-1236(86)90015-7

[22] Nielsen, M., and I. Chuang *Quantum Computation and Quantum Information*, Cambridge University Press, Cambridge, 2000. https://doi.org/10.1017/CBO9780511976667

[23] Paoli, F., A. Ledda, R. Giuntini, and H. Freytes, On some properties of quasi-MV algebras and $\sqrt{\prime}$ quasi-MV algebras, *Reports on Mathematical Logic* 44: 31–63, 2009.

[24] Paoli, F., A. Ledda, M. Spinks, H. Freytes, and R. Giuntini, Logics from $\sqrt{\prime}$ quasi-MV algebras, *International Journal of Theoretical Physics* 50(12): 3882–3902, 2011. https://doi.org/10.1007/s10773-011-0865-0

A. Dvurečenskij, O. Zahiri
Mathematical Institute
Slovak Academy of Sciences
Štefánikova 49
814 73 Bratislava
Slovakia
dvurecen@mat.savba.sk

O. Zahiri
om.zahiri@gmail.com

A. Dvurečenskij
Faculty of Sciences
Palacký University Olomouc
tř. 17. listopadu 12
771 46 Olomouc
Czech Republic

and

Department of Mathematics
Constantine the Philosopher University in Nitra
Tr. A. Hlinku 1
949 01 Nitra
Slovakia